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TRANSIENT AERODYNAMIC BEHAVIOR OF AN AIRFOIL DUE TO DIFFERENT ARBITRARY MODES OF NONSTATIONARY MOTIONS IN A SUPERSONIC FLOW

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The investigation of drag-characteristic behavior of the wing in steady flow is omitted entirely from this paper, because it is not a difficult matter for a designer to calculate the drag characteristics with the information contained in this paper and available solutions of the steady case.

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TRANSIENT REACTION OF AN AIRFOIL DUE TO UNIT-STEP CHANGE OF FLAPPING AND VERTICAL GUST

As shown in references 1 and 6, the linearized partial differential equation of the irrotational flow of a compressible nonviscous fluid is

$$\phi_{xx} + \phi_{yy} = \frac{1}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\phi \tag{1}$$

where ϕ is the disturbance potential of the flow, the x-axis is positive in the opposite direction of the flight, and the y-axis is positive in the upward direction. The origin is attached to the airfoil leading edge. The details of the notations are given in appendix A. In the supersonic case, the disturbance potential of such an airfoil is equivalent to that due to a source sheet at the $y = 0$ plane with time-dependent strength and is

$$\phi(x,y,t) = -\frac{1}{\pi}\frac{1}{\left(M^2 - 1\right)^{1/2}}\int_0^{\xi_1}\int_{\tau_1}^{\tau_2}\frac{v(\xi,+0,t - \tau)}{\sqrt{\left(\tau - \tau_1\right)\left(\tau_2 - \tau\right)}}\,d\tau\,d\xi \tag{2}$$

where $v(\xi,+0,t - \tau)$ is the vertical velocity component on the wing surface at the point $(\xi,+0)$ at an earlier time $t - \tau$. The term $\dfrac{v(\xi,+0,t - \tau)}{\left(M^2 - 1\right)^{1/2}}$ is just the source-sheet strength per unit area. In addition,

$$\left.\begin{aligned} \tau_1 &= \frac{M(x - \xi)}{a\left(M^2 - 1\right)} - \frac{r}{a} \\ \tau_2 &= \frac{M(x - \xi)}{a\left(M^2 - 1\right)} + \frac{r}{a} \end{aligned}\right\} \tag{3}$$

where

$$\left.\begin{aligned} r &= \frac{1}{M^2 - 1}\sqrt{(x - \xi)^2 - \left(M^2 - 1\right)(y - \eta)^2} \\ \xi_1 &= x - y\sqrt{M^2 - 1} \\ &\left(\xi_1 > 0\right) \end{aligned}\right\} \tag{4}$$

If the point of interest is on the lower side of the wing surface $(x,-0,t)$ equation (2) reduces to

$$\phi(x,-0,t) = \frac{1}{\pi}\frac{1}{\left(M^2 - 1\right)^{1/2}}\int_0^x \int_{\tau_1}^{\tau_2} \frac{v(\xi,-0,t - \tau)}{\sqrt{\left(\tau - \tau_1\right)\left(\tau_2 - \tau\right)}}\, d\tau\, d\xi \tag{5}$$

With the above equation the transient case due to sudden change of angle of attack of the airfoil about any arbitrary point has been treated in reference 1. The important results of that paper are given in table 1 for use in the present development. Two additional interesting cases will be treated as follows.

Vertical Flapping

At time t_0, the airfoil suddenly descends with constant velocity $\bar{\dot{h}}$. The unit-step downwash at any point on the wing can be expressed in terms of Dirichlet's integral as

$$v(x,-0,t) = \dot{h}(t) = \dot{h}\left[\frac{1}{\pi}\int_0^\infty \frac{\sin \omega\left(t - t_0\right)}{\omega}\, d\omega + \frac{1}{2}\right] \tag{6}$$

where ω is the angular velocity per second or a frequency parameter. Figure 1(a) shows $\frac{\dot{h}(t)}{U} = \alpha_h(t)$. Substituting equation (6) into equation (5), there results, after manipulation,

$$\phi_h(x,-0,t) = -\frac{\bar{\dot{h}}}{\pi^2\left(M^2 - 1\right)^{1/2}}\left[\int_0^x d\xi \int_{\tau_1}^{\tau_2} d\tau \int_0^\infty \frac{\sin \omega\left(t - \tau - t_0\right)}{\omega\sqrt{\left(\tau - \tau_1\right)\left(\tau_2 - \tau\right)}}\, d\omega + \frac{\pi}{2}\int_0^x d\xi \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{\left(\tau - \tau_1\right)\left(\tau_2 - \tau\right)}}\right]$$

$$= -\frac{\bar{\dot{h}}}{\pi\left(M^2 - 1\right)^{1/2}}\int_0^x d\xi \int_0^\xi \frac{d\omega}{\omega} \sin \omega\left(t - t_0 - M\lambda\right)J_0(\omega\lambda) - \frac{\bar{\dot{h}}x}{2\left(M^2 - 1\right)^{1/2}} \qquad (7)$$

or after integration the potential is expressed in three zones according to the value of $\frac{x}{U\left(t - t_0\right)} = \nu$ as follows:

$$\phi_h(x,-0,t) = -\frac{\bar{\dot{h}}x}{\pi}\begin{cases} \frac{1}{\left(M^2 - 1\right)^{1/2}} \sin^{-1}\left(\frac{m}{\nu} - M\right) - \frac{1}{M\nu}\left[\sin^{-1} M(1 - \nu) - \frac{\pi}{2}\right] & \left(\frac{M - 1}{M} \leqq \nu \leqq \frac{M + 1}{M}\right) \\ \frac{-\pi}{\left(M^2 - 1\right)^{1/2}} & \left(0 \leq \nu \leqq \frac{M - 1}{M}\right) \\ -\frac{\pi}{M\nu} & \left(\frac{M + 1}{M} \leqq \nu\right) \end{cases} \qquad (8)$$

where

$$\nu = \frac{x}{U(t - t_0)} = \frac{x'}{(t' - t_0')} \qquad x' = \frac{x}{C}$$

$$m = \frac{M^2 - 1}{M} \qquad t' = \frac{tU}{C}$$

$$\lambda = \frac{x}{a(M^2 - 1)} \qquad t_0' = \frac{t_0 U}{C}$$

and $J_0(\omega\lambda)$ is the Bessel function of the first kind of the zero order. Table 2 gives the detailed characteristics of $\frac{\phi_h}{\bar{\alpha}_h}$, $\frac{C_{ph}}{\bar{\alpha}_h} = f_h$, $\frac{C_{Lh}}{\bar{\alpha}_h} = F_h$, and $\frac{C_{Mh}}{\bar{\alpha}_h} = G_h$ as functions of time. Figures 2 to 4 show these characteristics. The detailed methods of calculation are quite similar to those given in reference 1, and are omitted in this paper.

In figures 2 to 4, a few important features may be explained. Within the zone III $\left(0 \leqq \frac{t' - t_0'}{x'} \leqq \frac{M}{M + 1}\right)$, $f_h = \frac{C_{ph}}{\bar{\alpha}_h}$ is inversely proportional to Mach number and independent of time and the location of x'. But in time zone I $\left(\frac{M}{M - 1} \leqq \frac{t' - t_0'}{x'}\right)$, $f_h = \frac{C_{ph}}{\bar{\alpha}_h}$ is inversely proportional to $\sqrt{M^2 - 1}$ which corresponds to the steady case of Ackeret's result. In zone II $\left(\frac{M}{M + 1} \leqq \frac{t' - t_0'}{x'} \leqq \frac{M}{M - 1}\right)$, $\frac{C_{ph}}{\bar{\alpha}_h}$ varies not with x' or $t' - t_0'$ alone, but with the conical coordinate $\frac{t' - t_0'}{x'}$. It is a complicated function of M and $\frac{t' - t_0'}{x}$. Thus the present solution is similar to Busemann's conical flow. So is the behavior of $F_h = \frac{C_{Lh}}{\bar{\alpha}_h}$ against $t' - t_0'$.

As far as $G_h = \frac{C_{Mh}}{\bar{\alpha}_h}$ is concerned, it varies both with $1/M$ and $t' - t_0'$ in zone I. In general, f_h, F_h, and G_h increase with decreasing Mach number, if $\frac{t' - t_0'}{x'}$ or $t' - t_0'$ is kept constant.

Vertical Gust

At time t, if the airfoil begins to encounter a uniform vertical gust of velocity $\dot{g}$, such unit-step downwash $\dot{g}$ at any point on the airfoil can also be expressed in terms of Dirichlet's integral

$$v(x,-0,t) = \dot{g}(t) = \bar{\dot{g}}\left[\frac{1}{\pi}\int_0^\infty \frac{\sin \omega\left(t - t_0 - \frac{x}{U}\right)}{\omega} d\omega + \frac{1}{2}\right] \tag{9}$$

Figure 1(b) shows $\frac{\dot{g}(t)}{U} = \alpha_g(t)$ as a function of both t and x.

Substituting equation (9) into equation (5), the disturbance potential ϕ_g due to vertical gust can be written as

$$\phi_g(x,-0,t) = -\frac{\bar{\dot{g}}}{\pi^2\left(M^2 - 1\right)^{1/2}}\left[\int_0^x d\xi \int_0^{\tau_2} d\tau \int_0^\infty \frac{\sin \omega\left(t - \tau - t_0 - \frac{\xi}{U}\right)}{\omega\sqrt{\left(\tau - \tau_1\right)\left(\tau_2 - \tau\right)}} d\omega - \frac{\pi}{2}\int_0^x d\xi \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{\left(\tau - \tau_1\right)\left(\tau_2 - \tau\right)}}\right]$$

$$= \frac{-\bar{\dot{g}}}{\pi\left(M^2 - 1\right)^{1/2}}\int_0^x d\xi \int_0^\infty \frac{d\omega}{\omega} \sin \omega\left(t - t_0 - \frac{\xi}{U} - M\lambda\right)J_0(\omega\lambda) - \frac{x\bar{\dot{g}}}{2\left(M^2 - 1\right)^{1/2}} \tag{10}$$

or, after integration, ϕ_g has to be also expressed in three zones,

$$\phi_g(x,-0,t) = -\frac{\bar{\dot{g}}x}{\pi}\begin{cases} \dfrac{1}{\left(M^2 - 1\right)^{1/2}}\left[\dfrac{\pi}{2} + \sin^{-1}\left(\dfrac{m}{\nu} - M\right)\right] & \left(\dfrac{M - 1}{M} \leqq \nu \leqq \dfrac{M + 1}{M}\right) \\ \dfrac{\pi}{\left(M^2 - 1\right)^{1/2}} & \left(0 \leqq \nu \leqq \dfrac{M - 1}{M}\right) \\ 0 & \left(\dfrac{M + 1}{M} \leqq \nu\right) \end{cases} \tag{11}$$

It is interesting to note that ϕ_h and ϕ_g are similar in form. The Fourier integral method of Bessel functions used will take care of the three zones automatically without considering each zone individually. This is one of the main advantages of the present method.

Table 3 shows the detailed characteristics of $\dfrac{\phi_g}{\bar{\alpha}_g}$, $\dfrac{C_{pg}}{\bar{\alpha}_g} = f_g$, $\dfrac{C_{Lg}}{\bar{\alpha}_g} = F_g$, and $\dfrac{C_{Mg}}{\bar{\alpha}_g} = G_g$ as functions of time. Figures 5 to 7 show f_g, F_g, and G_g as functions of time. The detailed calculations are also omitted.

Pitching

For convenience, the essential results of reference 1 for the pitching case are given in table 1. Figures 8 to 10 show some characteristics of this case.

In the case where $x_0' = 0$ or the airfoil rotates about the leading edge at a constant rate $\left(\dot{\alpha}_1 - \dot{\alpha}_0\right)$ starting at $t' - t_0'$, $f_{\dot{\alpha}}' = \dfrac{C_{p\dot{\alpha}}}{x'\left(\dot{\alpha}_1 - \dot{\alpha}_0\right)}$ varies linearly with $\dfrac{t' - t_0'}{x'}$ but inversely with M in zone III $\left(0 \leqq \dfrac{t' - t_0'}{x'} \leqq \dfrac{M}{M + 1}\right)$ if the angle of attack is small at any instant.

In zone I $\left(\frac{M}{M - 1} \leqq \frac{t' - t_0'}{x'}\right)$ it also varies linearly with $\frac{t' - t_0'}{x'}$, but inversely with $\sqrt{M^2 - 1}$. If $M > 1.5$, $\frac{C_{p\dot{\alpha}}}{x'(\dot{\alpha}_1 - \dot{\alpha}_0)}$ also varies approximately linearly with $\frac{t' - t_0'}{x'}$ and is continuous in value with zones I and III although the exact expression as shown in table 1 is quite complicated. Of course, as the Mach number is near to 1, this linear approximation becomes worse. It seems better to use the exact solution in the table. Similar statements can be made about $F_{\dot{\alpha}} = \frac{C_{L\dot{\alpha}}}{\left(\bar{\dot{\alpha}}_1 - \bar{\dot{\alpha}}_0\right)}$ and $G_{\dot{\alpha}} = \frac{C_{M\dot{\alpha}}}{\left(\bar{\dot{\alpha}}_1 - \bar{\dot{\alpha}}_0\right)}$.

If $x_0' \neq 0$ some additional contributions have to be considered as

$$f_{\dot{\alpha}} = \frac{C_{p\dot{\alpha}}}{x'\left(\bar{\dot{\alpha}}_1 - \bar{\dot{\alpha}}_0\right)} = f_{\dot{\alpha}}' - \Delta f_{\dot{\alpha}}' \tag{12}$$

$$F_{\dot{\alpha}} = \frac{C_{L\dot{\alpha}}}{\left(\bar{\dot{\alpha}}_1 - \bar{\dot{\alpha}}_0\right)} = F_{\dot{\alpha}}' - \Delta F_{\dot{\alpha}}' \tag{13}$$

$$G_{\dot{\alpha}} = \frac{C_{M\dot{\alpha}}}{\left(\bar{\dot{\alpha}}_1 - \bar{\dot{\alpha}}_0\right)} = G_{\dot{\alpha}}' - \Delta G_{\dot{\alpha}}' \tag{14}$$

where $\frac{\Delta f_{\dot{\alpha}}'}{x_0'} = f_h$, $\frac{\Delta F_{\dot{\alpha}}'}{x_0'} = F_h$, and $\frac{\Delta G_{\dot{\alpha}}'}{x_0'} = G_h$ are given in table 2 and figures 2 to 4.

REACTION ON AN AIRFOIL DUE TO MOTION OF AN ARBITRARY TIME-DEPENDENT FUNCTION

In reference 1 it has been pointed out that, if the solution of a linear, ordinary or partial, differential equation with a simple boundary condition is known and if the given boundary condition can be obtained by superposition of the simple boundary condition, the general solution of the same differential equation can then be obtained by superimposing the elementary solution corresponding to the simple boundary. In the case of the initial-value problem, the statement is also true. With the transient problem, the important relation of superposition is the convolution integral. This integral relation is associated with many names in modern mathematical physics. In the transient problem of heat transfer, it is called Duhamel's integral in honor of the French mathematician Duhamel (reference 7, pp. 403-404). In England it is commonly known as Borel's theorem (reference 8, pp. 321-328). In this country it is commonly known as Carson's integral, particularly in electrical transmission. In Germany it is usually called the Faltung theorem (reference 9, pp. 159-167), and this was known to Tricomi. The concept of the convolution integral plays a very important part in applied mathematics.

In aerodynamics, Jones applied this integral to the problem of airplane dynamics in incompressible flow (reference 10). The results may be considered natural in his problem of ordinary differential equations. Garrick (reference 11) applied this integral to find the relation of the Wagner Function to the Theodorsen function in the case of two-dimensional incompressible nonstationary flow.

In reference 1, the present author constructed an integral relation entirely from the physical concept because in this problem of supersonic transient flow the kernel function is discontinuous in the first derivative at a certain instant of time. The ordinary concept of the convolution integral cannot be applied without detailed examination, although the result can be rewritten in quite a similar form to the ordinary convolution integral. For example, the pressure coefficient $C_p(t',x')$, the lift coefficient $C_L(t')$, and the moment coefficient $C_M(t')$ for any arbitrary $\alpha(t) = \frac{\dot{h}(t)}{U}$ or $\frac{\dot{g}(t)}{U}$ can be obtained by means of the convolution integral of the kernel function due to unit-step change in angle of attack as follows:

$$\begin{Bmatrix} C_p(t',x') \\ C_L(t') \\ C_M(t') \end{Bmatrix} = \begin{Bmatrix} f_i(t',x') \\ F_i(t') \\ G_i(t') \end{Bmatrix} \left[\alpha_1(+0) - \alpha_0(-0)\right] + \int_0^{t'} d\tau'\dot{\alpha}(\tau) \begin{Bmatrix} f(t' - \tau',x') \\ F(t' - \tau') \\ G(t' - \tau') \end{Bmatrix} \tag{15}$$

where F, G, and f are kernel functions discontinuous in first derivatives as shown in the following table.

Function \ Condition	$\frac{M}{M-1} \leqq t'$	$\frac{M}{M+1} \leqq t' \leqq \frac{M}{M-1}$	$0 \leqq t' \leqq \frac{M}{M+1}$
$f(t',x')$	$f_1(t',x')$	$f_2(t',x')$	$f_3(t',x')$
$F(t')$	$F_1(t')$	$F_2(t')$	$F_3(t')$
$G(t')$	$G_1(t')$	$G_2(t')$	$G_3(t')$

The subscript i should be equal to 1, 2, or 3 according to the above range of time in the term of the left-hand side. Owing to the discontinuity of these functions, the integral on the right must be broken up into parts according to the range of time for each function. For example, in the flapping case, the lift is:

(a) For $\frac{M}{M-1} \leqq t'$,

$$C_L(t')_I = \left[\alpha_h(+0) - \alpha_h(-0)\right] F_1(t') + \int_0^{t' - \frac{M}{M-1}} \dot{\alpha}_h(\tau')F_1(t' - \tau')\, d\tau' + \int_{t' - \frac{M}{M-1}}^{t' - \frac{M}{M+1}} \dot{\alpha}_h(\tau')F_2(t' - \tau')\, d\tau' + \int_{t' - \frac{M}{M+1}}^{t'} \dot{\alpha}_h(\tau')F_3(t' - \tau')\, d\tau' \tag{16}$$

(b) For $\frac{M}{M+1} \leqq t' \leqq \frac{M}{M-1}$,

$$C_L(t')_{II} = \left[\alpha_h(+0) - \alpha_h(-0)\right] F_2(t') + \int_0^{t' - \frac{M}{M+1}} \dot{\alpha}_h(\tau')F_2(t' - \tau')\, d\tau' +$$

$$\int_{t'-\frac{M}{M+1}}^{t'} \dot{\alpha}_h(\tau')F_3(t' - \tau')\, d\tau' \tag{17}$$

(c) For $0 \leqq t' \leqq \frac{M}{M+1}$,

$$C_L(t')_{III} = \left[\alpha_h(+0) - \alpha_h(-0)\right] F_3(t') + \int_0^{t'} \dot{\alpha}_h(\tau')F_3(t' - \tau')\, d\tau' \tag{18}$$

Tables 2 and 3 give the expressions for f's, F's, and G's in the case of flapping and vertical gusts. In the case of pitching, the above relation is the same as equation (15), except that α should be replaced by $\dot{\alpha}$ as given in reference 1 except for C_p which is replaced by $\frac{C_p}{x'}$ in equation (15). Another complication in this pitching case is the location of the axis of rotation x_0', which is the ratio of the axis location from the leading edge divided by the chord. The contribution due to the nonzero value of x_0' is denoted by $\Delta f_{\dot{\alpha}}$, $\Delta F_{\dot{\alpha}}$, and $\Delta G_{\dot{\alpha}}$ and should be added respectively to $f_{\dot{\alpha}}$, $F_{\dot{\alpha}}$, and $G_{\dot{\alpha}}$. As expected, $\frac{\Delta f_{\dot{\alpha}}}{x_0'} = f_h$, $\frac{\Delta F_{\dot{\alpha}}}{x_0'} = F_h$, and $\frac{\Delta G_{\dot{\alpha}}}{x_0'} = G_h$ as shown in figures 2, 3, and 4, respectively.

As an application of the above convolution integrals, the cases of the harmonically oscillating airfoil starting abruptly from rest at $t = 0$ have been investigated as follows:

Pitching oscillation about the leading edge $(x_0' = 0)$.- In the case of pitching oscillation about the leading edge,

$$\left.\begin{aligned} \alpha(t') &= \bar{\alpha} && \text{for } t' < 0 \\ \alpha(t') &= \bar{\alpha}e^{i\omega t} = \bar{\alpha}e^{i\beta t'} && \text{for } t' > 0 \end{aligned}\right\} \qquad (19)$$

where $\frac{\omega}{2\pi}$ is the frequency of oscillation, $\beta = \frac{\omega C}{U}$, and $t' = \frac{tU}{C}$. (See table 4.) Actually, equation (19) is a complex quantity, and the real part is

$$\left.\begin{aligned} \text{Real } \alpha(t') &= \bar{\alpha} && \text{for } t' < 0 \\ \text{Real } \alpha(t') &= \bar{\alpha}\cos\beta t' && \text{for } t' > 0 \end{aligned}\right\} \qquad (20)$$

The expression $\alpha(t') = \bar{\alpha}$ for $t' < 0$ is necessary in order that $\dot{\alpha}(t')$ be finite at $t' = 0$, because only finite $\dot{\alpha}(t')$ is allowed for pitching. (For details see reference 1.) The imaginary part is

$$\left.\begin{aligned} \text{Imaginary } \alpha(t') &= 0 && \text{for } t' < 0 \\ \text{Imaginary } \alpha(t') &= \bar{\alpha}\sin\beta t' && \text{for } t' > 0 \end{aligned}\right\} \qquad (21)$$

With the complex $\alpha(t')$, the analysis is more convenient than with the real or the imaginary part alone. If the axis of rotation is not at the leading edge $x_0' \neq 0$, the additional effect can be obtained from the flapping case.

Flapping oscillation.- In the case of flapping oscillation,

$$\left.\begin{aligned} \alpha_h(t) &= 0 && \text{for } t < 0 \\ \alpha_h(t) &= \frac{\dot{h}}{U} = \bar{\alpha}_h e^{i\omega t} && \text{for } t > 0 \end{aligned}\right\} \qquad (22)$$

The solution is given in table 5 for the three time zones. Some of the integrals are given in appendix B.

Harmonically oscillating gust.- In the case of a harmonically oscillating gust,

$$\left.\begin{aligned} \alpha_g(t) &= 0 && t < 0 \\ \alpha_g(t) &= \bar{\alpha}_g e^{i\omega t} && t > 0 \end{aligned}\right\} \tag{23}$$

The solution is given in table 6. Some of the integrals are given in appendix B. All three cases are essential to the supersonic flutter. The present analysis gives the aerodynamic behavior of the airfoil for the whole time history, if the harmonic oscillation in flapping and pitching start abruptly at $t = 0$. If $t' > \frac{M}{M - 1}$, the present work should check with reference 6 exactly. Appendix C shows the comparison.

To show the present analysis graphically, C_L and C_M have been calculated with $M = 1.5$ and $\beta = \pi/2$ for the three oscillations. Figures 11 and 12 show the C_L and C_M of the pitching oscillation. Figures 13 and 14 show them for flapping and figures 15 and 16 show them for vertical gust. The corresponding values of $\alpha(t')$ are also shown for comparison of phase shifts. The maximum C_L and C_M are larger in the transient beginning than the steady case which is represented by dotted curves. The transient effect dies out completely when $t' > \frac{M}{M - 1}$. It is interesting to note that the extremes (maximum or minimum) of C_L or C_M always lead the corresponding extremes of α. But in the case of flapping oscillation and oscillating gust, the extremes of C_L or C_M always lag behind the corresponding extremes of α. As the supersonic Mach number nears 1, the transient effect becomes more pronounced and lasts longer.

In the time zone III $\left(0 \leqq t' \leqq \frac{M}{M + 1}\right)$, the expressions for $C_L(t')$ and $C_M(t')$ are very simple in all three cases of harmonic oscillation for pitching, flapping, and vertical gust. (Refer to tables 4, 5, and 6.) In the time zone II $\left(\frac{M}{M + 1} \leqq t' \leqq \frac{M}{M - 1}\right)$, the expressions are rather complicated and cannot be represented in a closed form. Three new functions or integrals have to be defined. Let

$$J_0\left(\frac{\beta}{m},\theta_1\right) = \frac{1}{\pi}\int_{\theta_1}^{\pi} d\theta \, \exp\left(\frac{i\beta}{m}\cos\theta\right) \tag{24}$$

where $m = \frac{M^2 - 1}{M}$, $\beta > 0$, $M > 1$, $0 \leqq \theta_1 \leqq \pi$, and θ_1 is usually a function of time t'. As designated by Garrick and Rubinow, $J_0\left(\frac{\beta}{m},\theta_1\right)$ may be called the incomplete Bessel function of the zero order. It is complex except when $\theta_1 = 0$. At $\theta_1 = 0$, equation (24) reduces to the ordinary Bessel function of the zero order $J_0\left(\frac{\beta}{m}\right)$ which is real. Similarly, the incomplete Bessel function of the first order can be defined as

$$J_1\left(\frac{\beta}{m},\theta_1\right) = -\frac{i}{\pi}\int_{\theta_1}^{\pi} d\theta \, \cos\theta \, \exp\left(\frac{i\beta}{m}\cos\theta\right) \tag{25}$$

which is also complex, but reduces to $J_1\left(\frac{\beta}{m}\right)$, the ordinary Bessel function of the first order when $\theta_1 = 0$. The new integral is now defined as

$$C(\beta,M;\ \theta_1) = \frac{1}{\pi}\int_{\theta_1}^{\pi} d\theta \, \exp\left(\frac{-i\beta M}{M - \cos\theta}\right) \tag{26}$$

When $\theta_1 = 0$, it reduces to

$$C(\beta,M) = \frac{1}{\pi}\int_{0}^{\pi} d\theta \, \exp\left(\frac{-i\beta M}{M - \cos\theta}\right) \tag{27}$$

For convenience, $C(\beta,M)$ is called the C-function and $C(\beta,M;\ \theta_1)$ is called the incomplete C-function. The incomplete C-function occurs in time zone II for all three cases. In time zone III, it reduces to $C(\beta,M)$. This function, to the author's knowledge, has never been explored before and is investigated in appendix D. In time zone III, $C_L(t')$ and $C_M(t')$ can be expressed with the known functions

except $C(\beta,M)$ in all the three cases. It is expected that $C(\beta,M)$ is as important to supersonic flutter as the Theodorsen function is to flutter in incompressible flow.

TRANSIENT AERODYNAMIC BEHAVIOR OF THE CONTROL SURFACE

Since the principle of superposition holds for the two-dimensional linear problem if the deflection angle δ of the control surface is measured from the main wing, the control surface itself may be considered as an independent airfoil at the corresponding deflection angle δ and with its chord C_S. Under such a consideration, the lift and moment of the control surface itself in nonstationary motion can be obtained directly from the result of the airfoil in the early section, if the proper time scale is used. The time required to travel a length C_S is $t^* = \frac{C_S}{U}$, and, if t^* is used as the time unit, the nondimensional time $t'' = \frac{t}{t^*} = \frac{Ut}{C_S}$ is connected with the true time and nondimensional time $t' = \frac{t}{\bar{t}} = \frac{Ut}{C}$ in the relation

$$\left(t'' - t_0''\right)t^* = t = t'\bar{t}$$

and

$$\frac{t^*}{\bar{t}} = \frac{C_S}{C} = k_c \tag{28}$$

where $t_0''t^*$ is the difference in starting time of the control movement from that of the main wing. Thus for an individual increment in $\dot{\delta}$ as the step function at time t_i'', the lift and moment coefficients of the control surface at later times are

$$\left.\begin{aligned} C_L(t'') &= \Delta\dot{\delta}_i\left(t_i''\right)F_j\left(t'' - t_i''\right) \\ C_{Mx_i}(t'') &= \Delta\dot{\delta}_i\left(t_i''\right)G_j\left(t'' - t_i''\right) \end{aligned}\quad (j = 1,2,3)\right\} \tag{29}$$

where F_j and G_j are the same as shown in table 1 except that t' is replaced by t''.

The lift coefficient of the control surface alone is

$$c_L(t'') = \Delta\dot{\delta}_0(0)F_j(t'') + \int_0^{t'' - \frac{M}{M-1}} \ddot{\delta}\left(t_i''\right)F_1\left(t'' - t_i''\right) dt_i'' +$$

$$\int_{t'' - \frac{M}{M-1}}^{t'' - \frac{M}{M+1}} \ddot{\delta}\left(t_i''\right)F_2\left(t'' - t_i''\right) dt_i'' +$$

$$\int_{t'' - \frac{M}{M+1}}^{t''} \ddot{\delta}\left(t_i''\right)F_3\left(t'' - t_i''\right) dt_i'' \tag{30}$$

Then, the increment of the total wing lift coefficient due to the control surface is

$$\Delta C_L = k_c c_L\left(\frac{t'}{k_c}\right)$$

$$= k_c\left[\Delta\dot{\delta}_0(0)F_j\left(\frac{t'}{k_c}\right) + \int_0^{\frac{t'}{k_c}} \ddot{\delta}(\tau')F_j\left(\frac{t' - \tau'}{k_c}\right) d\,\frac{\tau'}{k_c}\right]$$

where

$$\left.\begin{aligned}
F_j\left(\frac{t'}{k_c}\right) &= F_1\left(\frac{t'}{k_c}\right) & & 0 \leqq \frac{t'}{k_c} \leqq t'' \\
F_j\left(\frac{t'}{k_c}\right) &= F_2\left(\frac{t'}{k_c}\right) & & \frac{t'' - \frac{M}{M - 1}}{k_c} \leqq \frac{t'}{k_c} \leqq \frac{t'' - \frac{M}{M + 1}}{k_c} \\
F_j\left(\frac{t'}{k_c}\right) &= F_3\left(\frac{t'}{k_c}\right) & & \frac{t'' - \frac{M}{M + 1}}{k_c} \leqq \frac{t'}{k_c}
\end{aligned}\right\} \tag{31}$$

Similarly, the increment of total moment coefficient due to the control surface can be found.

As an example of the effect of the control surface on the total lift, the following calculation is made.

(1) $M = 1.5$ $\quad a = 1000$ ft/sec $\quad t_1 = 0.02$ sec $\quad (t_1' = 5)$

$C = 6$ ft $\quad C_S = 2$ ft $(k_c = 1/3)$ $\quad x_0 = 0$ $\quad x_1 = 4$ ft

(2) Operation schedule of the main wing:

$$\alpha = \dot{\alpha} = 0 \qquad t' < 0$$

$$\alpha = \bar{\dot{\alpha}} \qquad 0 \leqq t' \leqq t_1'$$

$$\dot{\alpha} = 0 \qquad t_1' < t'$$

(3) Operation schedule of the control surface:

$$\delta(t') = \dot{\delta}(t') = 0 \qquad t' < 0$$

$$\dot{\delta}(t') = 2\bar{\dot{\alpha}} \qquad 0 \leqq t' \leqq \frac{t_1'}{3}$$

$$\dot{\delta}(t') = 0 \qquad \frac{t_1'}{3} \leqq t' \leqq \frac{2t_1'}{3}$$

$$\dot{\delta}(t') = -2\bar{\dot{\alpha}} \qquad \frac{2t_1'}{3} \leqq t' \leqq t_1'$$

$$\delta(t') = \dot{\delta}(t') = 0 \qquad t' > t_1'$$

The distribution of $\dot{\alpha}(t')$ and $\dot{\delta}(t')$ against t' and the geometry of the airfoil and control surface are given in figure 17. The contribution of the control surface to total lift, that is, $\Delta C_L(t')$, is shown in the lower curve; $\overline{C}_L(t')$ of the main wing is shown in the dotted curve which has been given in reference 1. Since the ratio of $\dot{\delta}(t')$ and $\dot{\alpha}(t')$ is given, this curve can be used for any arbitrary $\bar{\dot{\alpha}}$. In this curve $C_{L0} = \dfrac{4\dot{\alpha}t_1}{(M^2 - 1)^{1/2}}$ is used instead of $\bar{\dot{\alpha}}$.

DISCUSSION

For the airfoil in a flow of constant supersonic speed, the transient effect due to pitching, flapping, or vertical gust will damp out in a time period $\frac{C}{U - a}$ immediately after the change in angle of attack ceases. Although the result is obtained from the linearized theory, it is expected to be approximately true for the nonlinear theory if the angle of attack is reasonably small. After that time period, the lift, wave drag, and moment become the same as given by Ackeret in steady two-dimensional linearized supersonic flow. The transient effect on C_p, C_L, and C_M becomes more pronounced and lasts longer as the supersonic Mach number approaches 1. The same is also true of the transient period of the harmonic oscillations in pitching, flapping, or vertical gust which start from rest at $t = 0$.

In the case of pitching with constant rate, $\frac{C_{p\dot{\alpha}}}{x'\left(\dot{\alpha}_1 - \dot{\alpha}_0\right)}$, $\frac{C_{L\dot{\alpha}}}{\left(\dot{\alpha}_1 - \dot{\alpha}_0\right)}$, and $\frac{C_{M\dot{\alpha}}}{\left(\dot{\alpha}_1 - \dot{\alpha}_0\right)}$ can be approximated satisfactorily with a straight line in the time zone II, if $M > 1.5$. When such an approximation is adopted, the case of pitching harmonic oscillation can be evaluated very quickly. It is also easy to evaluate any arbitrary motion in pitching. In the present analysis, the exact expressions in table 3 were used instead of the above approximation because no simple approximation can be obtained for either flapping or vertical gust.

In solving the transient problems with the convolution integral, one new function $C(\beta,M)$ is discovered. For the time from the abrupt start $\left(t > \frac{M}{M - 1} \frac{C}{U}\right)$, C_p, C_L, and C_M in the cases of harmonic oscillations have to be expressed in terms of $C(\beta,M)$. It seems of comparable importance to supersonic flutter as the Theodorsen function is to flutter in incompressible flow. In the transient time zone II, where $\frac{M}{M + 1} \frac{C}{U} \leqq t \leqq \frac{M}{M - 1} \frac{C}{U}$, a new function $C\left(\beta,M;\ \theta_1\right)$, called the incomplete C-function, occurs which assumes an importance equal to that of the incomplete Bessel functions. More complete calculation of $C(\beta,M)$ seems useful for the analysis of supersonic flutter in order to cover wider ranges of Mach number and frequency than those given in reference 6. Appendix D gives some of the properties of $C(\beta,M)$, $C\left(\beta,M;\ \theta_1\right)$, and the incomplete Bessel functions.

The pressure coefficients $\dfrac{C_{p\dot{\alpha}}}{x'(\dot{\alpha}_1 - \dot{\alpha}_0)}$ in pitching, $\dfrac{C_{ph}}{\bar{\alpha}_h}$ in flapping, and $\dfrac{C_{pg}}{\bar{\alpha}_g}$ in vertical gust are functions of Mach number and a conical parameter $\dfrac{t' - t_0'}{x'}$ only. They are analogous to the behavior of Busemann's conical flow. As pointed out in reference 1, these results can be applied to a yawing infinite wing, if the leading edge is ahead of the Mach line.

The effects of additional degrees of freedom such as the control surface or servoflaps can be evaluated with the result of the pitching and flapping of the main wing as shown under Transient Aerodynamic Behavior of the Control Surface. With the present basic approach, a good aeronautical engineer with ingenuity should be able to solve all two-dimensional problems of flutter and any other arbitrary motion. Of course in the case of complicated time-dependent functions of angle of attack some numerical or graphical integration of the convolution integral might be necessary as shown in references 1 and 10.

For the case of the harmonic oscillation with constant maximum amplitude with abrupt start at $t = 0$ the absolute magnitudes of the extremes in C_L and C_M for pitching and flapping degrees of freedom are larger in the transient region $\dfrac{M}{M + 1}\dfrac{C}{U} \leqq t \leqq \dfrac{M}{M - 1}\dfrac{C}{U}$ than those in the steady region $\dfrac{M}{M - 1}\dfrac{C}{U} < t$, particularly as the supersonic Mach number nears 1. In addition, the maximum and minimum of the oscillating angle of attack are not in phase with the maximum and minimum of C_L or C_M. In the case of pitching, the load leads the angle of attack; in the case of flapping, the angle of attack leads the load. Also, the angle of attack leads the load in the case of vertical gust.

As an interesting example of the harmonic oscillation building up to flutter or damping out, the case with complex ω should be investigated. With the convolution integral, it seems within the reach of the present analysis. Of course, the imaginary part of ω must be determined from the interaction of aerodynamic forces and the elastic behavior of the wing.

The Johns Hopkins University
Baltimore, Md., November 4, 1949

APPENDIX A

SYMBOLS

a	velocity of sound
c	wing chord
C_L	transient lift coefficient
C_{LO}	two-dimensional lift coefficient in steady case
C_M	transient moment coefficient about axis of rotation
C_p	pressure coefficient $\left(2(p - p_1)/\rho_1 u_1^2\right)$
c_s	chord of control surface

$$C(\beta,M) = \frac{1}{\pi}\int_0^{\pi} d\theta \exp\left(\frac{-i\beta}{1 - \frac{\cos\theta}{M}}\right) = \frac{1}{\pi}\int_0^{\pi} d\theta \exp\left(\frac{-i\beta M}{M - \cos\theta}\right)$$

$$\bar{C}(\sigma,M) = \frac{1}{\pi}\int_0^{\pi} d\theta \exp\left(\frac{-i\sigma}{M - \cos\theta}\right)$$

$$C(\beta,M;\ \theta_1) = \frac{1}{\pi}\int_{\theta_1}^{\pi} d\theta \exp\left(\frac{-i\beta M}{M - \cos\theta}\right)$$

$F(\ )$	kernel function concerning lift coefficient
$f(\ )$	kernel function concerning pressure coefficient
$G(\ )$	kernel function concerning moment coefficient
$\dot{g}$	velocity of vertical gust
$\bar{\dot{g}}$	maximum velocity of uniform vertical gust

$\dot{h}$	descending velocity
$\bar{\dot{h}}$	maximum velocity of vertical flapping wing
$J_0(\ )$	Bessel function of zero order
$J_1(\ )$	Bessel function of first order
$k_c = \frac{C_S}{C} = \frac{t^*}{\bar{t}}$	
M	Mach number
m	parameter of Mach number $\left(\frac{M^2 - 1}{M}\right)$
t	time
$\bar{t} = C/U$	
t'	nondimensional time $(t/\bar{t})$
$t^* = C_S/U$	
t"	nondimensional time (control surface) (t/t^*)
U	free-stream velocity
v	y-component of velocity
x	axis along chord direction
x'	percent of chord (x/C)
x_0	axis of rotation of entire airfoil
y	vertical axis
α	angle of attack
$\bar{\alpha}$	constant angle of attack
$\bar{\alpha}_g = \bar{\dot{g}}/U$	
$\bar{\alpha}_h = \bar{\dot{h}}/U$	

β	frequency parameter for oscillating airfoils
δ	deflection angle of control surface
η	source location along y-axis
θ	running variable
θ_1	lower limit of θ

$\lambda = x/a\left(M^2 - 1\right)$

ν	$x/U\left(t - t_0\right) = x'/\left(t' - t_0'\right)$
ξ	source location along x-axis

$\sigma = \beta M$

τ	time interval
τ'	nondimensional time interval $(\tau/\bar{t})$
ϕ	velocity potential
ω	frequency parameter (angular velocity per second)

Subscripts:

g	vertical gust
h	flapping
$\dot{\alpha}$	pitching (changing angle of attack)
δ	control surface

APPENDIX B

SUMMARY OF INTEGRALS

Important Integrals in the Case of Steady Harmonic Oscillation

The following integrals are used in evaluating C_L and C_M for the time zone $t' \geq \frac{M}{M-1}$ (as shown in tables 4, 5, and 6).

$$I_1 \int_{t'-\frac{M}{M-1}}^{t'-\frac{M}{M+1}} e^{i\beta\tau'} \left\{1 + \frac{2}{\pi}\sin^{-1}\left[m(t'-\tau') - M\right]\right\} d\tau' = \frac{-2}{i\beta} e^{i\beta\left(t'-\frac{M}{M-1}\right)} + \frac{2e^{i\beta\left(t'-\frac{M}{m}\right)}}{i\beta} J_0\left(\frac{\beta}{m}\right)$$

$$I_2 \int_{t'-\frac{M}{M-1}}^{t'-\frac{M}{M+1}} e^{i\beta\tau'}(t'-\tau') \left\{1 + \frac{2}{\pi}\sin^{-1}\left[m(t'-\tau') - M\right]\right\} d\tau' = \frac{-2}{(i\beta)^2} e^{i\beta\left(t'-\frac{M}{M-1}\right)}\left(i\beta\frac{M}{M-1} + 1\right) +$$

$$\frac{e^{i\beta\left(t'-\frac{M}{m}\right)}}{i\beta}\left[\frac{2M}{m} J_0\left(\frac{\beta}{m}\right) + \frac{2}{i\beta} J_0\left(\frac{\beta}{m}\right) - \frac{2i}{m} J_1\left(\frac{\beta}{m}\right)\right]$$

$$I_3 \int_{t'-\frac{M}{M-1}}^{t'-\frac{M}{M+1}} e^{i\beta\tau'} \sqrt{(t'-\tau')^2 - M^2(t'-\tau'-1)^2}\, d\tau' = \frac{M\pi i}{i\beta\sqrt{M^2-1}} e^{i\beta\left(t'-\frac{M}{m}\right)} J_1\left(\frac{\beta}{m}\right)$$

$$I_4 \int_{t'-\frac{M}{M-1}}^{t'-\frac{M}{M+1}} e^{i\beta\tau'}(t'-\tau') \sqrt{(t'-\tau')^2 - M^2(t'-\tau'-1)^2}\, d\tau' = \frac{M\pi i}{i\beta\sqrt{M^2-1}} e^{i\beta\left(t'-\frac{M}{m}\right)}\left[\frac{i}{m} J_0\left(\frac{\beta}{m}\right) +\right.$$

$$\left.\left(\frac{M}{m} - \frac{2i}{\beta}\right) J_1\left(\frac{\beta}{m}\right)\right]$$

$$I_5 \int_{t'-\frac{M}{M-1}}^{t'-\frac{M}{M+1}} e^{i\beta\tau'} \cos^{-1} M \frac{t'-\tau'-1}{t'-\tau'}\, d\tau' = \frac{e^{i\beta t'}}{i\beta}\left[\pi e^{-i\beta\frac{M}{M+1}} - \int_0^{\pi} e^{-i\left(\frac{\beta M}{M-\cos\theta}\right)} d\theta\right]$$

$$I_6 \int_{t' - \frac{M}{M-1}}^{t' - \frac{M}{M+1}} e^{i\beta\tau'}(t' - \tau') \cos^{-1} M \frac{t' - \tau' - 1}{t' - \tau'} d\tau' = \frac{e^{i\beta t'}}{i\beta}\left[\frac{M\pi}{M + 1} e^{-i\beta\frac{M}{M+1}} - \right.$$

$$\left. \frac{\pi M}{\sqrt{M^2 - 1}} e^{-i\beta\frac{M}{m}} J_0\left(\frac{\beta}{m}\right)\right] + \frac{1}{i\beta} I_5$$

$$I_7 \int_{t' - \frac{M}{M-1}}^{t' - \frac{M}{M+1}} e^{i\beta\tau'}(t' - \tau')^2 \cos^{-1} M \frac{t' - \tau' - 1}{t' - \tau'} d\tau' = \frac{e^{i\beta t'}}{i\beta}\left[\pi\left(\frac{M}{M + 1}\right)^2 e^{-i\beta\frac{M}{M+1}} - \right.$$

$$\left. \frac{\pi M^3}{\left(M^2 - 1\right)^{3/2}} e^{-i\beta\frac{M}{m}} J_0\left(\frac{\beta}{m}\right) + \frac{\pi M^2 i}{\left(M^2 - 1\right)^{3/2}} e^{-i\beta\frac{M}{m}} J_1\left(\frac{\beta}{m}\right)\right] + \frac{2}{i\beta} I_6$$

$$I_8 \int_{t' - \frac{M}{M-1}}^{t' - \frac{M}{M+1}} e^{i\beta\tau'}(t' - \tau')^3 \cos^{-1} M \frac{t' - \tau' - 1}{t' - \tau'} d\tau' = \frac{e^{i\beta t'}}{i\beta}\left\{\pi\left(\frac{M}{M + 1}\right)^3 e^{-i\beta\frac{M}{M+1}} - \right.$$

$$\left. \frac{\pi M^3 e^{-i\beta\frac{M}{m}}}{\left(M^2 - 1\right)^{5/2}}\left[\left(M^2 + 1\right)J_0\left(\frac{\beta}{m}\right) - \left(2Mi + \frac{m}{\beta}\right)J_1\left(\frac{\beta}{m}\right)\right]\right\} + \frac{3}{i\beta} I_7$$

$$I_9 \int_{t' - \frac{M}{M-1}}^{t' - \frac{M}{M+1}} e^{i\beta\tau'}(t' - \tau')^2 \sqrt{(t' - \tau')^2 - M^2(t' - \tau' - 1)^2}\, d\tau' = \frac{M^3}{\left(M^2 - 1\right)^{5/2}} e^{i\beta\left(t' - \frac{M}{m}\right)}$$

$$\left\{\frac{\pi i}{i\beta}\left(M^2 + 1\right) + \frac{4\left(M^2 - 1\right)}{i\beta} + 6\left(\frac{m}{i\beta}\right)^2 J_1\left(\frac{\beta}{m}\right) - \frac{\pi}{i\beta}\left(2M + \frac{3m}{i\beta}\right)J_0\left(\frac{\beta}{m}\right)\right\}$$

Important Integrals in the Case of Transient Harmonic Oscillation

The following integrals are used in calculating C_L and C_M for the time zone $\frac{M}{M+1} \leqq t' \leqq \frac{M}{M-1}$ (as shown in tables 4, 5, and 6).

$$I_1' \int_0^{t' - \frac{M}{M+1}} e^{i\beta\tau'} \left\{1 + \frac{2}{\pi} \sin^{-1} \left[m(t' - \tau') - M\right]\right\} d\tau' = -\frac{2}{i\beta} + \frac{2}{\pi i\beta} \cos^{-1} (mt' - M) +$$

$$\frac{2}{\pi} \frac{e^{i\beta\left(t' - \frac{M}{m}\right)}}{i\beta} \int_{\cos^{-1}(mt'-M)}^{\pi} e^{-i\beta\frac{\cos \phi}{m}} d\phi$$

$$I_2' \int_0^{t' - \frac{M}{M+1}} e^{i\beta\tau'}(t' - \tau') \left\{1 + \frac{2}{\pi} \sin^{-1} \left[m(t' - \tau') - M\right]\right\} d\tau' = \left[\frac{t'}{i\beta} + \frac{1}{(i\beta)^2}\right]\left[\frac{2}{\pi} \cos^{-1} (mt' - M) - 2\right] +$$

$$\frac{2e^{i\beta\left(t' - \frac{M}{m}\right)}}{\pi i\beta} \int_{\cos^{-1}(mt'-M)}^{\pi} \left(\frac{M}{m} + \frac{1}{i\beta} + \frac{\cos \phi}{m}\right) e^{-i\beta\frac{\cos \phi}{m}} d\phi$$

$$I_3' \int_0^{t' - \frac{M}{M+1}} e^{i\beta\tau'} \sqrt{(t' - \tau')^2 - M^2(t' - \tau' - 1)^2}\, d\tau' = \frac{M}{\sqrt{M^2 - 1}} \frac{-1}{i\beta} \sqrt{1 - (mt' - M)^2} -$$

$$\frac{Me^{i\beta\left(t' - \frac{M}{m}\right)}}{\sqrt{M^2 - 1}\, i\beta} \int_{\cos^{-1}(mt'-M)}^{\pi} \cos \phi\, e^{-i\beta\frac{\cos \phi}{m}} d\phi$$

$$I_4' \int_0^{t' - \frac{M}{M+1}} e^{i\beta\tau'}(t' - \tau')\sqrt{(t' - \tau')^2 - M^2(t' - \tau' - 1)^2}\, d\tau' = \frac{M}{\sqrt{M^2 - 1}}\left[\frac{-t'}{i\beta} - \frac{2}{(i\beta)^2}\right]\sqrt{1 - (mt' - M)^2} -$$

$$\frac{M}{\sqrt{M^2 - 1}} \frac{e^{i\beta\left(t' - \frac{M}{m}\right)}}{i\beta} \int_{\cos^{-1}(mt' - M)}^{\pi} \left(\frac{1}{m} + \frac{M\cos\phi}{m} + \frac{2\cos\phi}{i\beta}\right) e^{-i\beta\frac{\cos\phi}{m}}\, d\phi$$

$$I_5' \int_0^{t' - \frac{M}{M+1}} \left(e^{i\beta\tau'}\right) \cos^{-1} M\frac{t' - \tau' - 1}{t' - \tau'}\, d\tau' = \frac{\pi}{i\beta} e^{i\beta\left(t' - \frac{M}{M+1}\right)} - \frac{e^{i\beta t'}}{i\beta} \int_{\cos^{-1}\frac{M(t'-1)}{t'}}^{\pi} e^{-i\frac{\beta M}{M - \cos\theta}}\, d\theta -$$

$$\frac{1}{i\beta}\cos^{-1}\frac{M(t' - 1)}{t'}$$

$$I_6' \int_0^{t' - \frac{M}{M+1}} e^{i\beta\tau'}(t' - \tau')\cos^{-1} M\frac{t' - \tau' - 1}{t' - \tau'}\, d\tau' = \frac{\pi}{i\beta}\frac{M}{M + 1} e^{i\beta\left(t' - \frac{M}{M+1}\right)} - \frac{t'}{i\beta}\cos^{-1}\frac{M(t' - 1)}{t'} -$$

$$\frac{M}{\sqrt{M^2 - 1}} \frac{e^{i\beta\left(t' - \frac{M}{m}\right)}}{i\beta} \int_{\cos^{-1}(mt' - M)}^{\pi} e^{-i\beta\frac{\cos\phi}{m}}\, d\phi + \frac{1}{i\beta} I_5'$$

$$I_7' \int_0^{t' - \frac{M}{M+1}} e^{i\beta\tau'}(t' - \tau')^2\cos^{-1} M\frac{t' - \tau' - 1}{t' - \tau'}\, d\tau' = \frac{\pi}{i\beta}\left(\frac{M}{M + 1}\right)^2 e^{i\beta\left(t' - \frac{M}{M+1}\right)} - \frac{t'^2}{i\beta}\cos^{-1}\frac{M(t' - 1)}{t'} -$$

$$\frac{M^2}{\left(M^2 - 1\right)^{3/2}} \frac{e^{i\beta\left(t' - \frac{M}{m}\right)}}{i\beta} \int_{\cos^{-1}(mt' - M)}^{\pi} (M + \cos\phi) e^{-i\beta\frac{\cos\phi}{m}}\, d\phi + \frac{2}{i\beta} I_6'$$

$$I_8' \int_0^{t'-\frac{M}{M+1}} e^{i\beta\tau'}(t' - \tau')^3 \cos^{-1} M \frac{t' - \tau' - 1}{t' - \tau'} d\tau' = \frac{\pi}{i\beta}\left(\frac{M}{M+1}\right)^3 e^{i\beta\left(t' - \frac{M}{M+1}\right)} - \frac{t'^3}{i\beta} \cos^{-1} \frac{M(t' - 1)}{t'} -$$

$$\frac{M^3}{\left(M^2 - 1\right)^{5/2}} \frac{e^{i\beta\left(t' - \frac{M}{m}\right)}}{i\beta} \int_{\cos^{-1}(mt'-M)}^{\pi} \left(M^2 + 1 + 2M \cos \phi + \frac{m}{i\beta} \cos \phi\right) e^{-i\beta\frac{\cos\phi}{m}} d\phi +$$

$$\frac{1}{i\beta} \frac{M^2}{\left(M^2 - 1\right)^{3/2}} \sqrt{1 - (mt' - M)^2} + \frac{3}{i\beta} I_7'$$

$$I_9' \int_0^{t'-\frac{M}{M+1}} e^{i\beta\tau'}(t' - \tau')^2 \sqrt{(t' - \tau')^2 - M^2(t' - \tau' - 1)^2}\, d\tau' = -\frac{M^3}{\left(M^2 - 1\right)^{5/2}} \frac{\sqrt{1 - (mt' - M)^2}}{i\beta}\left[m^2 t'^2 + \right.$$

$$\left.\frac{m}{i\beta}(M + 3mt') + 6\left(\frac{m}{i\beta}\right)^2\right] - \frac{M^3}{\left(M^2 - 1\right)^{5/2}} \frac{e^{i\beta\left(t' - \frac{M}{m}\right)}}{i\beta} \int_{\cos^{-1}(mt'-M)}^{\pi} \left\{2M + \frac{3m}{i\beta} + \cos\phi\left[M^2 + 1 + \frac{4Mm}{i\beta} + \right.\right.$$

$$\left.\left.6\left(\frac{m}{i\beta}\right)^2\right]\right\} e^{-i\beta\frac{\cos\phi}{m}} d\phi$$

APPENDIX C

CHECK WITH NACA TN 1158

To check the present results through convolution integrals with Garrick and Rubinow's work, a case of the steady harmonic oscillation of the angle of attack about the leading edge is investigated. With their notation, C_L due to the harmonic oscillation of the angle of attack alone $\left(\alpha = \alpha_0 e^{i\omega t}\right)$ may be verified from their equation (26) as

$$C_{L\alpha} = 4k^2 e^{i\omega t} \alpha_0 \left(L_3 + iL_4\right) \tag{C1}$$

At $x_0 = 0$,

$$L_3 = L_3' \qquad L_3 = L_3' - 2x_0 L_1$$

$$L_4 = L_4' \qquad L_4 = L_4' - 2x_0 L_2$$

At $M = 2$, for $\frac{1}{k} = 1.667$, from their table II,

$$L_3' = 1.50219 \qquad L_4' = 0.69968$$

Hence,

$$C_{L\alpha} = \alpha_0 (2.16315 + 1.00754i) e^{i\omega t}$$

NOTATION

NACA TN 1158	Present paper
t	$t' = \frac{tU}{C}$
ω	$\beta = \frac{\omega C}{U}$
b	$C = 2b$
v	$U = aM = v$
$k = \frac{\omega b}{v}$	$\beta = 2k$
$e^{i\omega t}$	$\exp\,(i\beta t') = \exp\,(i\omega t)$

In the present case $\alpha = \bar{\alpha}\exp\,(i\beta t')$. From table 4 for $t' \geqq \frac{M}{M-1}$,

$$\frac{C_L(t')}{\bar{\alpha}} = i\beta\exp\left[i\beta\left(t' - \frac{M^2}{M^2-1}\right)\right]\left\{J_0\left(\frac{\beta}{m}\right)\left[\frac{2}{\sqrt{M^2-1}} + \frac{2\left(2M^2-1\right)}{M^2\sqrt{M^2-1}}\,\frac{1}{i\beta}\right] + J_1\left(\frac{\beta}{m}\right)i\left[\frac{2}{M\sqrt{M^2-1}}\right]\left(1 + \frac{1}{i\beta}\right)\right\} + i\beta\exp\,(i\beta t')\left\{\left[\frac{2}{M} + \frac{4}{Mi\beta} + \frac{2}{M^3(i\beta)^2}\right]\left[1 - C(\beta,M)\right]\right\}$$

whence, at $M = 2$ and $\beta = 1.2$, $C_L = \bar{\alpha}\exp\,(i\beta t')(2.16315 + 1.00753i)$. The expressions are identical, within computational limits.

APPENDIX D

INVESTIGATION OF THE FEW IMPORTANT INTEGRALS RELATED TO

SUPERSONIC FLUTTER AND TRANSIENT PROBLEMS

C-Function

The problem is investigation of the C-function or the integral

$$C(\beta,M) = \frac{1}{\pi}\int_0^{\pi} d\theta \exp\left(\frac{-i\beta M}{M - \cos\theta}\right) \qquad \begin{matrix}(\beta > 0)\\ (M > 1)\end{matrix}$$

If the above integral is differentiated with respect to β and M,

$$\frac{\partial C}{\partial M} = \frac{i\beta}{\pi}\int_0^{\pi} d\theta \left[\exp\left(\frac{-i\beta M}{M - \cos\theta}\right)\right]\frac{\cos\theta}{(M - \cos\theta)^2} \tag{D1}$$

$$\frac{\partial C}{\partial \beta} = \frac{-i}{\pi}\int_0^{\pi} d\theta \left[\exp\left(\frac{-i\beta M}{M - \cos\theta}\right)\right]\frac{M}{M - \cos\theta} \tag{D2}$$

$$\frac{\partial^2 C}{\partial \beta^2} = \frac{-1}{\pi}\int_0^{\pi} d\theta \left[\exp\left(\frac{i\beta M}{M - \cos\theta}\right)\right]\frac{M^2}{(M - \cos\theta)^2} \tag{D3}$$

With the above three relations, it is found that $C(\beta,M) = C_R + iC_I$ satisfies the differential equation

$$\frac{\partial C}{\partial \beta} = i\frac{\partial^2 C}{\partial \beta^2} + \frac{M}{\beta}\frac{\partial C}{\partial M} \tag{D4}$$

If the real and imaginary parts of $C(\beta,M)$ are taken separately,

$$\left.\begin{aligned}\frac{\partial C_R}{\partial \beta} &= -\frac{\partial^2 C_I}{\partial \beta^2} + \frac{M}{\beta}\frac{\partial C_R}{\partial M}\\ \frac{\partial C_I}{\partial \beta} &= \frac{\partial^2 C_R}{\partial \beta^2} + \frac{M}{\beta}\frac{\partial C_I}{\partial M}\end{aligned}\right\} \tag{D5}$$

which are two simultaneous differential equations. The required six boundary conditions can be obtained from $C(\beta,M)$ as follows.

For $\beta = 0$,

$$\left.\begin{aligned}C_R(0,M) &= 1 \qquad & C_I(0,M) &= 0\\ \frac{\partial C_R}{\partial \beta} &= 0 \qquad & \frac{\partial C_I}{\partial \beta} &= \frac{-M}{\sqrt{M^2 - 1}}\end{aligned}\right\} \tag{D6}$$

For $M \longrightarrow \infty$,

$$C_R = \cos\beta \qquad C_I = -\sin\beta \tag{D7}$$

The integral $\frac{1}{\pi}\int_0^{\pi} d\theta \exp\left(\frac{-i\beta M}{M - \cos\beta}\right)$ can be written as

$$\bar{C}(\sigma,M) = \frac{1}{\pi}\int_0^{\pi} d\theta \exp\left(\frac{-i\sigma}{M - \cos\theta}\right)$$

where $\sigma = \beta M$. Under the new definition, the function $\bar{C}(\sigma,M)$ can be shown to satisfy the differential equation

$$i\sigma\frac{\partial^2 \bar{C}}{\partial \sigma^2} = \frac{\partial \bar{C}}{\partial M} \tag{D8}$$

or

$$
\left.
\begin{aligned}
\sigma \frac{\partial^2 \bar{C}_R}{\partial \sigma^2} &= \frac{\partial \bar{C}_I}{\partial M} \\
\sigma \frac{\partial^2 \bar{C}_I}{\partial \sigma^2} &= -\frac{\partial C_R}{\partial M}
\end{aligned}
\right\} \qquad (D9)
$$

It is easy to show that the real and imaginary parts of $\bar{C}(\sigma,M)$ must satisfy the equations

$$
\left.
\begin{aligned}
&\sigma \frac{\partial^2}{\partial \sigma^2}\left(\sigma \frac{\partial^2 \bar{C}_R}{\partial \sigma^2}\right) = \frac{\partial^2 \bar{C}_R}{\partial M^2} \qquad 0 \leq \sigma \leq \infty \\
\text{and} \qquad & \\
&\sigma \frac{\partial^2}{\partial \sigma^2}\left(\sigma \frac{\partial^2 \bar{C}_I}{\partial \sigma^2}\right) = \frac{\partial^2 \bar{C}_I}{\partial M^2} \qquad 1 \leq M \leq \infty
\end{aligned}
\right\} \qquad (D10)
$$

The corresponding boundary conditions are:

For $\sigma = 0, \quad M > 1,$

$$
\left.
\begin{aligned}
\bar{C}_R &= 1 & \qquad \bar{C}_I &= 0 \\
\frac{\partial \bar{C}_R}{\partial \sigma} &= 0 & \qquad \frac{\partial \bar{C}_I}{\partial \sigma} &= \frac{-1}{\sqrt{M^2 - 1}} \\
\frac{\partial^2 \bar{C}_R}{\partial \sigma^2} &= \frac{M}{\left(M^2 - 1\right)^{3/2}} & \qquad \frac{\partial^2 \bar{C}_I}{\partial \sigma^2} &= 0
\end{aligned}
\right\} \qquad (D11)
$$

For $M \longrightarrow \infty,$

$$
\left.
\begin{aligned}
\bar{C}_R &= 1 & \qquad \bar{C}_I &= 0 \\
\frac{\partial \bar{C}_R}{\partial M} &= 0 & \qquad \frac{\partial \bar{C}_I}{\partial M} &= 0 \\
\frac{\partial \bar{C}_R}{\partial \sigma} &= 0 & \qquad \frac{\partial \bar{C}_I}{\partial \sigma} &= 0
\end{aligned}
\right\} \qquad (D12)
$$

The preceding function is so important to the supersonic flutter problem that thorough investigation of the function and the other solutions of the corresponding differential equation seems advisable. To show the general behavior of this new function, a case with $M = 1.3$ and $0 \leqq \beta \leqq 3$ has been calculated numerically. See table 7 and figure 18.

Incomplete C-function

The incomplete C-function is

$$C\left(\beta,M;\ \theta_1\right) = \frac{1}{\pi}\int_{\theta_1}^{\pi} \exp\left(\frac{-i\beta M}{M - \cos\theta}\right) d\theta \tag{D13}$$

It can be shown that this incomplete C-function also satisfies the differential equations (D5) except the boundary conditions are obtained from equation (D13).

For $\beta = 0$,

$$\left.\begin{aligned} C_R &= \left(1 - \frac{\theta_1}{\pi}\right) & C_I &= 0 \\ \frac{\partial C_R}{\partial \beta} &= 0 & \frac{\partial C_I}{\partial \beta} &= \frac{-M}{\sqrt{M^2 - 1}}\left[1 - \frac{2}{\pi}\tan^{-1}\left(\sqrt{\frac{M + 1}{M - 1}}\tan\frac{\theta_1}{2}\right)\right] \end{aligned}\right\} \tag{D14}$$

For $M \longrightarrow \infty$,

$$C_R = \left(1 - \frac{\theta_1}{\pi}\right)\cos\beta \qquad C_I = -\left(1 - \frac{\theta_1}{\pi}\right)\sin\beta \tag{D15}$$

To show the nature of the incomplete C-function, a case with $M = 1.5$, $\beta = \frac{\pi}{2}$, and $0 \leqq \theta_1 \leqq \pi$ has been calculated by means of numerical integration. See table 8 and figure 19. The above data are used in the transient behavior as shown in figures 11 to 16.

Incomplete Bessel Functions

The incomplete Bessel functions are

$$J_0(z,\theta_1) = \frac{1}{\pi}\int_{\theta_1}^{\pi} \exp\,(+iz\cos\theta)\,d\theta \tag{D16}$$

$$J_1(z,\theta_1) = \frac{-i}{\pi}\int_{\theta_1}^{\pi} \cos\theta\,\exp\,(+iz\cos\theta)\,d\theta \tag{D17}$$

These two functions become $J_0(z)$ and $J_1(z)$, respectively, when $\theta_1 = 0$. They have been called the incomplete Bessel functions by Garrick and Rubinow. The case $M = 1.5$, $\beta = \frac{\pi}{2}$, and $0 \leqq \theta_1 \leqq \pi$ has been calculated by numerical methods. It is interesting that each has a real and an imaginary part. At $\theta_1 = 0$, both become real and ordinary Bessel functions. Both play as important roles as the incomplete C-function in harmonic oscillations at the transient time zone II. See table 8 and figures 20 and 21.

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TABLE 1

CHARACTERISTICS OF AN AIRFOIL WITH CHANGING ANGLE OF ATTACK $\bar{\alpha}$[1]

[$x_0' = 0$; see figs. 8, 9, and 10. For $x_0' \neq 0$ refer to table 2 and figs. 2, 3, and 4.]

	Zone I $\frac{M}{M-1} \leq \frac{t'-t_0'}{x'}$	Zone II $\frac{M}{M+1} \leq \frac{t'-t_0'}{x'} \leq \frac{M}{M-1}$	Zone III $0 \leq \frac{t'-t_0'}{x'} \leq \frac{M}{M+1}$
$\frac{\phi_{\dot{\alpha}}(x',-0,t')}{c^2(\bar{\alpha}_1-\bar{\alpha}_0)}$	$\frac{-(x')^2}{(M^2-1)^{1/2}}\left[1-\frac{1}{2(M^2-1)}\right]$	$\frac{-(x')^2}{(M^2-1)^{1/2}}\left(\left[\frac{t'-t_0'}{x'}-\frac{1}{2(M^2-1)}\right]\frac{1}{\pi}\left\{\frac{\pi}{2}+\sin^{-1}\left[\frac{m(t'-t_0')}{x'}-M\right]\right\}+\right.$ $\frac{(M^2-1)^{1/2}}{M}\left[\frac{t'-t_0'}{\pi x'}\cos^{-1}M\left(1-\frac{x'}{t'-t_0'}\right)+\right.$ $\left.\left.\frac{(t'-t_0')^2}{2\pi(x')^2}\left(\frac{1}{M}+\frac{x'}{m(t'-t_0')}\right)\sqrt{1-M^2\left(1-\frac{x'}{t'-t_0'}\right)^2}\right]\right)$	$-\frac{x'}{M}(t'-t_0')$
$\frac{C_{p\dot{\alpha}}(x',-0,t')}{x'(\bar{\alpha}_1-\bar{\alpha}_0)} = f_{\dot{\alpha}}$	$\frac{2}{(M^2-1)^{1/2}}\left(\frac{t'-t_0'}{x'}+\frac{M^2-2}{M^2-1}\right) = f_{1\dot{\alpha}}$	$\frac{2}{(M^2-1)^{1/2}}\left(\left(\frac{t'-t_0'}{x'}+\frac{M^2-2}{M^2-1}\right)\frac{1}{\pi}\left\{\frac{\pi}{2}+\sin^{-1}\left[\frac{m(t'-t_0')}{x'}-M\right]\right\}+\right.$ $\frac{(M^2-1)^{1/2}}{\pi M}\left(\frac{t'-t_0'}{x'}+1\right)\cos^{-1}M\left(1-\frac{x'}{t'-t_0'}\right)+$ $\left.\frac{(2M^2-1)(t'-t_0')}{\pi M^2 x'(M^2-1)^{1/2}}\sqrt{1-M^2\left(1-\frac{x'}{t'-t_0'}\right)^2}\right) = f_{2\dot{\alpha}}$	$\frac{2}{M}\left(\frac{t'-t_0'}{x'}+1\right) = f_{3\dot{\alpha}}$
$t'-t_0'$	Zone I $\frac{M}{M-1} \leq t'-t_0'$	Zone II $\frac{M}{M+1} \leq t'-t_0' \leq \frac{M}{M-1}$	Zone III $0 \leq t'-t_0' \leq \frac{M}{M+1}$
$\frac{c_{L\dot{\alpha}}(t'-t_0')}{(\dot{\alpha}_1-\dot{\alpha}_0)} = F_{\dot{\alpha}}$	$\frac{4}{(M^2-1)^{1/2}}\left[t'-t_0'+\frac{M^2-2}{2(M^2-1)}\right] = F_{1\dot{\alpha}}$	$\frac{4}{(M^2-1)^{1/2}}\left(\left[\frac{t'-t_0'}{2}+\frac{M^2-2}{4(M^2-1)}\right]\frac{2}{\pi}\left\{\frac{\pi}{2}+\sin^{-1}\left[m(t'-t_0')-M\right]\right\}+\right.$ $\frac{(M^2-1)^{1/2}}{M\pi}\left[t'-t_0'+\frac{(t'-t_0')^2}{4M^2}+\frac{1}{2}\right]\cos^{-1}M\left(\frac{t'-t_0'-1}{t'-t_0'}\right)+$ $\left.\frac{(M^2-1)^{1/2}}{4M^2\pi}(t'-t_0')+\frac{5M^2-3}{M^2-1}\sqrt{(t'-t_0')^2-M^2(t'-t_0'-1)^2}\right) = F_{2\dot{\alpha}}$	$\frac{1}{M^3}(t'-t_0')^2+\frac{4}{M}(t'-t_0')+$ $\frac{2}{M} = F_{3\dot{\alpha}}$
$\frac{c_{M\dot{\alpha}}(t'-t_0')}{(\dot{\alpha}_1-\dot{\alpha}_0)} = G_{\dot{\alpha}}$	$\frac{4}{(M^2-1)^{1/2}}\left[\frac{t'-t_0'}{2}+\frac{M^2-2}{3(M^2-1)}\right] = G_{1\dot{\alpha}}$	$\frac{4}{(M^2-1)^{1/2}}\left(\left[\frac{t'-t_0'}{2}+\frac{M^2-2}{3(M^2-1)}\right]\frac{1}{\pi}\left\{\frac{\pi}{2}+\sin^{-1}\left[m(t'-t_0')-M\right]\right\}+\right.$ $\frac{(M^2-1)^{1/2}}{M\pi}\left[\frac{(t'-t_0')^3}{12M^2}+\frac{t'-t_0'}{2}+\frac{1}{3}\right]\cos^{-1}M\left(\frac{t'-t_0'-1}{t'-t_0'}\right)+$ $\frac{(M^2-1)^{1/2}}{9M^2\pi}\left[\frac{M^2-4}{4M^2}(t'-t_0')^2+\right.$ $\left.\left.\frac{(t'-t_0')}{4}+\frac{7M^2-4}{M^2-1}\right]\sqrt{(t'-t_0')^2-M^2(t'-t_0'-1)^2}\right) = G_{2\dot{\alpha}}$	$\frac{(t'-t_0')^3}{3M^3}+\frac{2(t'-t_0')}{M}+$ $\frac{4}{3M} = G_{3\dot{\alpha}}$

[1]From reference 1.

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TABLE 2

CHARACTERISTICS OF A FLAPPING AIRFOIL WITH $\bar{h}\ (t > 0)$ OR ADDITIONAL CONTRIBUTION FOR PITCHING IN THE CASE $x_0' \neq 0$

$\left[\bar{\alpha}_h = \frac{\bar{h}}{U}.\right.$ See figs. 2, 3, and 4.$\left.\right]$

	Zone I $\frac{M}{M-1} \leqq \frac{t' - t_0'}{x'}$	Zone II $\frac{M}{M+1} \leqq \frac{t' - t_0'}{x'} \leqq \frac{M}{M-1}$	Zone III $0 \leqq \frac{t' - t_0'}{x'} \leqq \frac{M}{M+1}$
$\frac{\phi_h(x',-0,t')}{CU\bar{\alpha}_h}$	$\frac{-x'}{(M^2-1)^{1/2}}$	$\frac{-x'}{\pi(M^2-1)^{1/2}}\left\{\frac{\pi}{2} + \sin^{-1}\left[\frac{m(t'-t_0')}{x'} - M\right] - \frac{(M^2-1)^{1/2}(t'-t_0')}{Mx'}\cos^{-1} M\left(1 - \frac{x'}{t'-t_0'}\right)\right\}$	$\frac{-(t'-t_0')}{M}$
$\frac{C_{p_h}(x',-0,t')}{\bar{\alpha}_h} = f_h$ $= \frac{\Delta f_{\dot{\alpha}}}{x_0'}$	$\frac{2}{(M^2-1)^{1/2}} = f_{1h}$ $= \frac{\Delta f_{1\dot{\alpha}}}{x_0'}$	$\frac{2}{\pi}\left(\frac{1}{(M^2-1)^{1/2}}\left\{\frac{\pi}{2} + \sin^{-1}\left[\frac{m(t'-t_0')}{x'} - M\right]\right\} + \frac{1}{M}\cos^{-1} M\left(1 - \frac{x'}{t'-t_0'}\right)\right) = f_{2h} = \frac{\Delta f_{2\dot{\alpha}}}{x_0'}$	$\frac{2}{M} = f_{3h} = \frac{\Delta f_{3\dot{\alpha}}}{x_0'}$
$t' - t_0'$	Zone I $\frac{M}{M-1} \leqq t' - t_0'$	Zone II $\frac{M}{M+1} \leqq t' - t_0' \leqq \frac{M}{M-1}$	Zone III $0 \leqq t' - t_0' \leqq \frac{M}{M+1}$
$\frac{C_{L_h}(t'-t_0')}{\bar{\alpha}_h} = F_h$ $= \frac{\Delta F_{\dot{\alpha}}}{x_0'}$	$\frac{4}{(M^2-1)^{1/2}} = F_{1h}$ $= \frac{\Delta F_{1\dot{\alpha}}}{x_0'}$	$\frac{4}{(M^2-1)^{1/2}}\left(\frac{1}{\pi}\left\{\frac{\pi}{2} + \sin^{-1}\left[m(t'-t_0') - M\right]\right\} + \frac{(M^2-1)^{1/2}}{M\pi}\cos^{-1} M\frac{t'-t_0'-1}{t'-t_0'} + \right.$ $\left.\frac{(M^2-1)^{1/2}}{M^2\pi}\sqrt{(t'-t_0')^2 - M^2(t'-t_0'-1)^2}\right) = F_{2h} = \frac{\Delta F_{2\dot{\alpha}}}{x_0'}$	$\frac{4}{M} = F_{3h}$ $= \frac{\Delta F_{3\dot{\alpha}}}{x_0'}$
$\frac{C_{M_h}(t'-t_0')}{\bar{\alpha}_h} = G_h$ $= \frac{\Delta G_{\dot{\alpha}}}{x_0'}$	$\frac{2}{(M^2-1)^{1/2}} = G_{1h}$ $= \frac{\Delta G_{1\dot{\alpha}}}{x_0'}$	$\frac{4}{(M^2-1)^{1/2}}\left(\frac{1}{2\pi}\left\{\frac{\pi}{2} + \sin^{-1}\left[m(t'-t_0') - M\right]\right\} + \frac{(M^2-1)^{1/2}}{M\pi}\left[\frac{1}{2} - \frac{(t'-t_0')^2}{4M^2}\right]\cos^{-1} M\frac{t'-t_0'-1}{t'-t_0'} + \right.$ $\left.\frac{(M^2-1)^{1/2}}{4\pi M^2}(t'-t_0'+1)\sqrt{(t'-t_0')^2 - M^2(t'-t_0'-1)^2}\right) = G_{2h} = \frac{\Delta G_{2\dot{\alpha}}}{x_0'}$	$\frac{2}{M} - \frac{1}{M^3}(t'-t_0')^2 = G_{3h}$ $= \frac{\Delta G_{3\dot{\alpha}}}{x_0'}$

TABLE 3

CHARACTERISTICS OF AN AIRFOIL ENCOUNTERING A SHARP VERTICAL

GUST OF VELOCITY $\bar{\dot{g}}$

$\left[\bar{\alpha}_g = \frac{\bar{\dot{g}}}{U}.\right.$ See figs. 5, 6, and 7.$\left.\right]$

	Zone I $\frac{M}{M-1} \leqq \frac{t' - t_0'}{x'}$	Zone II $\frac{M}{M+1} \leqq \frac{t' - t_0'}{x'} \leqq \frac{M}{M-1}$	Zone III $0 \leqq \frac{t' - t_0'}{x'} \leqq \frac{M}{M+1}$
$\frac{\phi_g(x',-0,t')}{CU\bar{\alpha}_g}$	$\frac{-x'}{(M^2-1)^{1/2}}$	$\frac{-x'}{\pi(M^2-1)^{1/2}}\left\{\frac{\pi}{2} + \sin^{-1}\left[\frac{m(t'-t_0')}{x'} - M\right]\right\}$	0
$f_g = \frac{C_{p_g}(x',-0,t')}{\bar{\alpha}_g}$	$\frac{2}{(M^2-1)^{1/2}} = f_{1g}$	$\frac{2}{\pi(M^2-1)^{1/2}}\left\{\frac{\pi}{2} + \sin^{-1}\left[\frac{m(t'-t_0')}{x'} - M\right]\right\} = f_{2g}$	$0 = f_{3g}$
$t' - t_0'$	Zone I $\frac{M}{M-1} \leqq t' - t_0'$	Zone II $\frac{M}{M+1} \leqq t' - t_0' \leqq \frac{M}{M-1}$	Zone III $0 \leqq t' - t_0' \leqq \frac{M}{M+1}$
$\frac{C_{L_g}(t'-t_0')}{\bar{\alpha}_g} = F_g$	$\frac{4}{(M^2-1)^{1/2}} = F_{1g}$	$\frac{4}{(M^2-1)^{1/2}}\left(\frac{1}{\pi}\left\{\frac{\pi}{2} + \sin^{-1}\left[m(t'-t_0') - M\right]\right\} + \frac{(M^2-1)^{1/2}}{\pi M}(t'-t_0')\cos^{-1} M\left(\frac{t'-t_0'-1}{t'-t_0'}\right)\right) = F_{2g}$	$\frac{4}{M}(t'-t_0') = F_{3g}$
$\frac{C_{M_g}(t'-t_0')}{\bar{\alpha}_g} = G_g$	$\frac{2}{(M^2-1)^{1/2}} = G_{1g}$	$\frac{2}{(M^2-1)^{1/2}}\left(\frac{1}{\pi}\left\{\frac{\pi}{2} + \sin^{-1}\left[m(t'-t_0') - M\right]\right\} + \frac{(M^2-1)^{1/2}}{\pi M}(t'-t_0')^2\cos^{-1} M\left(\frac{t'-t_0'-1}{t'-t_0'}\right) - \frac{(M^2-1)^{1/2}}{M^2\pi}(t'-t_0')\sqrt{(t'-t_0')^2 - M^2(t'-t_0'-1)^2}\right) = G_{2g}$	$\frac{2}{M}(t'-t_0')^2 = G_{3g}$

TABLE 4

CHANGING ANGLE OF ATTACK

$$\left[\begin{matrix}\alpha = \bar{\alpha}e^{i\beta t'} & t' \geqq 0 \\ \alpha = \bar{\alpha} & 0 \geqq t'\end{matrix} \quad \text{in the case } x_0' = 0. \text{ See figs. 11 and 12.}\right]$$

	Zone I $\frac{M}{M-1} \leqq t'$	Zone II $\frac{M}{M+1} \leqq t' \leqq \frac{M}{M-1}$	Zone III $t' \leqq \frac{M}{M+1}$
$\frac{C_L(t')}{\bar{\alpha}}$	$i\beta e^{i\beta t'}\left[\frac{2}{M} + \frac{4}{Mi\beta} + \frac{2}{M^3(i\beta)^2}\right]\left(1 - \frac{1}{\pi}\int_0^{\pi} e^{-i\beta\frac{M}{M-\cos\theta}}\,d\theta\right)^{*} +$ $\frac{2i\beta}{(M^2-1)^{1/2}}\, e^{i\beta\left(t' - \frac{M}{m}\right)}\left(1 + \frac{2M^2-1}{M^2 i\beta}\right)J_0\left(\frac{\beta}{m}\right) -$ $\frac{2\beta e^{i\beta\left(t' - \frac{M}{m}\right)}}{(M^2-1)^{1/2}M}\left(1 + \frac{1}{i\beta}\right)J_1\left(\frac{\beta}{m}\right)$	$i\beta e^{i\beta t'}\left[\frac{2}{M} + \frac{4}{Mi\beta} + \frac{2}{M^3(i\beta)^2}\right]\left[1 - \frac{1}{\pi}\int_{\cos^{-1} M\left(\frac{t'-1}{t'}\right)}^{\pi} e^{-i\beta\frac{M}{M-\cos\theta}}\,d\theta\right]^{**} +$ $\frac{2i\beta}{\pi(M^2-1)^{1/2}}\, e^{i\beta\left(t' - \frac{M}{m}\right)}\int_{\cos^{-1} mt'-M}^{\pi}\left[1 + \frac{2M^2-1}{M^2 i\beta} +\right.$ $\left.\cos\phi\left(-\frac{1}{M} - \frac{1}{Mi\beta}\right)\right]e^{-i\beta\frac{\cos\phi}{m}}\,d\phi^{\dagger} - \frac{4}{\pi(M^2-1)^{1/2}}\left[\frac{\pi}{2} + \sin^{-1}(mt' - M)\right] -$ $\frac{2}{M^2\pi}\sqrt{1-(mt'-M)^2} - \frac{2}{M^3\pi}\left(t' + 2M^2 + \frac{1}{i\beta}\right)\cos^{-1} M\left(\frac{t'-1}{t'}\right) + \frac{4}{(M^2-1)^{1/2}}$	$i\beta e^{i\beta t'}\left[\frac{2}{M} + \frac{4}{Mi\beta} + \frac{2}{M^3(i\beta)^2}\right] -$ $\frac{2}{M^3}\left(t' + 2M^2 + \frac{1}{i\beta}\right) +$ $\frac{4}{(M^2-1)^{1/2}}$
$\frac{C_M(t')}{\bar{\alpha}}$	$i\beta e^{i\beta t'}\left[\frac{4}{3M} + \frac{2}{Mi\beta} + \frac{2}{M^3(i\beta)^3}\right]\left(1 - \frac{1}{\pi}\int_0^{\pi} e^{-i\beta\frac{M}{M-\cos\theta}}\,d\theta\right)^{*} +$ $\frac{2i\beta e^{i\beta\left(t' - \frac{M}{m}\right)}}{(M^2-1)^{1/2}}\left[\frac{2}{3} + \frac{3M^2-2}{3M^2 i\beta} - \frac{1}{M^2(i\beta)^2}\right]J_0\left(\frac{\beta}{m}\right) -$ $\frac{2\beta e^{i\beta\left(t' - \frac{M}{m}\right)}}{M(M^2-1)^{1/2}}\left[\frac{2}{3} + \frac{1}{3i\beta} + \frac{M^2-4}{3M^2(i\beta)^2}\right]J_1\left(\frac{\beta}{m}\right)$	$i\beta e^{i\beta t'}\left[\frac{4}{3M} + \frac{2}{Mi\beta} + \frac{2}{M^3(i\beta)^3}\right]\left[1 - \frac{1}{\pi}\int_{\cos^{-1} M\left(\frac{t'-1}{t'}\right)}^{\pi} e^{-i\beta\frac{M}{M-\cos\theta}}\,d\theta\right]^{**} +$ $\frac{i\beta e^{i\beta\left(t' - \frac{M}{m}\right)}}{\pi(M^2-1)^{1/2}}\int_{\cos^{-1}(mt'-M)}^{\pi}\left[\frac{4}{3} + \frac{2(3M^2-2)}{3M^2 i\beta} - \frac{2}{M^2(i\beta)^2}\right]e^{-i\beta\frac{\cos\phi}{m}}\,d\phi^{\dagger} -$ $\frac{i\beta e^{i\beta\left(t' - \frac{M}{m}\right)}}{\pi M(M^2-1)^{1/2}}\int_{\cos^{-1}(mt'-M)}^{\pi}\left[\frac{4}{3} + \frac{2}{3i\beta} + \frac{2(M^2-4)}{3M^2(i\beta)^2}\right]\cos\phi\, e^{-i\beta\frac{\cos\phi}{m}}\,d\phi^{\dagger} +$ $i\beta\sqrt{1-(mt'-M)^2}\left[-\frac{t'(M^2-4)}{3M^3\pi(M^2-1)^{1/2}} - \frac{M^2-7}{9M\pi(M^2-1)^{3/2}} -\right.$ $\left.\frac{8(M^2-3)}{9M^3\pi(M^2-1)^{1/2}i\beta}\right] - \frac{1}{M^3\pi}\left[t'^2 + \frac{2t'}{i\beta} + 2M^2 + \frac{2}{(i\beta)^2}\right]\cos^{-1} M\left(\frac{t'-1}{t'}\right) -$ $\frac{2}{\pi(M^2-1)^{1/2}}\left[\frac{\pi}{2} + \sin^{-1}(mt' - M)\right] + \frac{2}{(M^2-1)^{1/2}}$	$i\beta e^{i\beta t'}\left[\frac{4}{3M} + \frac{2}{Mi\beta} + \frac{2}{M^3(i\beta)^3}\right] -$ $\frac{1}{M^3}\left[t'^2 + \frac{2t'}{i\beta} + 2M^2 + \frac{2}{(i\beta)^2}\right] +$ $\frac{2}{(M^2-1)^{1/2}}$

*C-function $C(\beta,M)$.
**Incomplete C-function $C(\beta,M;\theta_1)$; see appendix D.
†Incomplete Bessel functions; see appendix D.

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TABLE 5

FLAPPING WING

$\left[\alpha_h = \bar{\alpha}_h e^{i\beta t'}.\right.$ See figs. 13 and 14.$\left.\right]$

	Zone I $\frac{M}{M-1} \leqq t'$	Zone II $\frac{M}{M+1} \leqq t' \leqq \frac{M}{M-1}$	Zone III $t' \leqq \frac{M}{M+1}$
$\frac{c_L(t')}{\bar{\alpha}_h}$	$\frac{4e^{i\beta t'}}{M}\left(1 - \frac{1}{\pi}\int_0^{\pi} e^{-i\beta\frac{M}{M-\cos\theta}}\,d\theta\right)^{*} +$ $\frac{4e^{i\beta\left(t'-\frac{M}{m}\right)}}{M\left(M^2-1\right)^{1/2}}\left[MJ_0\left(\frac{\beta}{m}\right) + iJ_1\left(\frac{\beta}{m}\right)\right]$	$\frac{4e^{i\beta t'}}{M}\left[1 - \frac{1}{\pi}\int_{\cos^{-1} M\left(\frac{t'-1}{t'}\right)}^{\pi} e^{-i\beta\frac{M}{M-\cos\theta}}\,d\theta\right]^{**} +$ $\frac{4e^{i\beta\left(t'-\frac{M}{m}\right)}}{M\pi\left(M^2-1\right)^{1/2}}\int_{\cos^{-1} mt'-M}^{\pi}(M-\cos\phi)e^{-i\beta\frac{\cos\phi}{m}}\,d\phi^{\dagger}$	$\frac{4}{M}e^{i\beta t'}$
$\frac{c_M(t')}{\bar{\alpha}_h}$	$\frac{2e^{i\beta t'}}{M}\left[1 - \frac{1}{M^2(i\beta)^2}\right]\left(1 - \frac{1}{\pi}\int_0^{\pi} e^{-i\beta\frac{M}{M-\cos\theta}}\,d\theta\right)^{*} +$ $\frac{2e^{i\beta\left(t'-\frac{M}{m}\right)}}{\left(M^2-1\right)^{1/2}}\left(1 + \frac{1}{M^2 i\beta}\right)J_0\left(\frac{\beta}{m}\right) +$ $\frac{2ie^{i\beta\left(t'-\frac{M}{m}\right)}}{M\left(M^2-1\right)^{1/2}}\left(1 + \frac{1}{i\beta}\right)J_1\left(\frac{\beta}{m}\right)$	$\frac{2e^{i\beta t'}}{M}\left(1 - \frac{1}{M^2(i\beta)^2}\right)\left[1 - \frac{1}{\pi}\int_{\cos^{-1}\frac{M(t'-1)}{t'}}^{\pi} e^{-i\beta\frac{M}{M-\cos\theta}}\,d\theta\right]^{**} +$ $\frac{2e^{i\beta\left(t'-\frac{M}{m}\right)}}{\pi\left(M^2-1\right)^{1/2}}\left(1 + \frac{1}{M^2 i\beta}\right)\int_{\cos^{-1} mt'-M}^{\pi} e^{-i\beta\frac{\cos\phi}{m}}\,d\phi^{\dagger} -$ $\frac{2e^{i\beta\left(t'-\frac{M}{m}\right)}}{M\pi\left(M^2-1\right)^{1/2}}\left(1 + \frac{1}{i\beta}\right)\int_{\cos^{-1} mt'-M}^{\pi}\cos\phi\, e^{-i\beta\frac{\cos\phi}{m}}\,d\phi^{\dagger} +$ $\frac{2}{M^3\pi}\left[\frac{t'}{i\beta} + \frac{1}{(i\beta)^2}\right]\cos^{-1} M\left(\frac{t'-1}{t'}\right) - \frac{2}{\pi M^2(i\beta)}\sqrt{t'^2 - M^2(t'-1)^2}$	$\frac{2}{M}e^{i\beta t'}\left[1 - \frac{1}{M^2(i\beta)^2}\right] +$ $\frac{2}{M^3}\left[\frac{t'}{i\beta} + \frac{1}{(i\beta)^2}\right]$

*C-function (β,M).
**Incomplete C-function $C(\beta,M;\ \theta_1)$; see appendix D.
†Incomplete Bessel functions; see appendix D.

TABLE 6

VERTICAL GUST

$\left[\alpha_g = \bar{\alpha}_g e^{i\beta t'}.\ \text{See figs. 15 and 16.}\right]$

	Zone I $\frac{M}{M-1} \leqq t'$	Zone II $\frac{M}{M+1} \leqq t' \leqq \frac{M}{M-1}$	Zone III $t' \leqq \frac{M}{M+1}$
$\frac{C_L(t')}{\bar{\alpha}_g}$	$\frac{4e^{i\beta t'}}{Mi\beta}\left(1 - \frac{1}{\pi}\int_0^{\pi} e^{-i\beta\frac{M}{M-\cos\theta}}\,d\theta\right)^{*}$	$\frac{4e^{i\beta t'}}{Mi\beta}\left[1 - \frac{1}{\pi}\int_{\cos^{-1}M\left(\frac{t'-1}{t'}\right)}^{\pi} e^{-i\beta\frac{M}{M-\cos\theta}}\,d\theta\right]^{**} -$ $\frac{4}{Mi\beta\pi}\cos^{-1}M\left(\frac{t'-1}{t'}\right)$	$\frac{4\left(e^{i\beta t'} - 1\right)}{Mi\beta}$
$\frac{C_M(t')}{\bar{\alpha}_g}$	$\frac{4e^{i\beta t'}}{M(i\beta)^2}\left(1 - \frac{1}{\pi}\int_0^{\pi} e^{-i\beta\frac{M}{M-\cos\theta}}\,d\theta\right)^{*} +$ $\frac{4e^{i\beta\left(t'-\frac{M}{m}\right)}}{Mi\beta\left(M^2-1\right)^{1/2}}\left[-MJ_0\left(\frac{\beta}{m}\right) - iJ_1\left(\frac{\beta}{m}\right)\right]$	$\frac{4e^{i\beta t'}}{M(i\beta)^2}\left[1 - \frac{1}{\pi}\int_{\cos^{-1}M\left(\frac{t'-1}{t'}\right)}^{\pi} e^{-i\beta\frac{M}{M-\cos\theta}}\,d\theta\right]^{**} +$ $\frac{4}{M^2\pi i\beta}\sqrt{1 - (mt' - M)^2} +$ $\frac{4e^{i\beta\left(t'-\frac{M}{m}\right)}}{\pi Mi\beta\left(M^2-1\right)^{1/2}}\int_{\cos^{-1}mt'-M}^{\pi}(\cos\phi - M)e^{-i\beta\frac{\cos\phi}{m}}\,d\phi^{\dagger} -$ $\frac{4}{M\pi i\beta}\left(t' + \frac{1}{i\beta}\right)\cos^{-1}M\,\frac{t'-1}{t'}$	$\frac{4e^{i\beta t'}}{M(i\beta)^2} -$ $\frac{4}{Mi\beta}\left(t' + \frac{1}{i\beta}\right)$

*C-function.
**Incomplate C-function $C(\beta,M;\,\theta_1)$; see appendix D.
†Incomplete Bessel functions; see appendix D.

TABLE 7

VALUES OF C-FUNCTION $C(\beta,M)$ FOR $M = 1.3$

β	Real	Imaginary
0	1.0	0
.3	.84153	-.41796i
.5	.62101	-.56797i
.8	.32000	-.56962i
1.0	.21844	-.49844i
1.2	.19147	-.44854i
1.5	.17883	-.47712i
1.7	.12058	-.53423i
2.0	-.48081	-.56279i
2.2	-.16005	-.51154i
2.4	-.22697	-.42491i
2.5	-.23982	-.38123i
2.93	-.22265	-.29355i

TABLE 8

VALUES OF INCOMPLETE C-FUNCTION AND INCOMPLETE BESSEL FUNCTIONS

$\left[\beta = \pi/2, \quad M = 1.5\right]$

θ_1	$C(\beta,M;\ \theta_1)$		$J_0\left(\frac{\beta}{m},\theta_1\right)$		$J_1\left(\frac{\beta}{m},\theta_1\right)$	
0	-0.05339	-0.47148i	0.29055	0	0.58148	0
.2	-.04952	-.53493i	.30946	.06078i	.52110	.01880i
.4	-.02526	-.59317i	.32378	.12278i	.46198	.03249i
.6	.02500	-.63079i	.32891	.18614i	.40647	.03707i
.8	.08770	-.63493i	.32058	.24910i	.35837	.03088i
1.0	.14434	-.60720i	.29582	.30750i	.32204	.01573i
1.2	.18435	-.558l9i	.25417	.35536i	.30020	-.00289i
1.4	.20575	-.49848i	.19871	.38580i	.29172	-.01572i
1.6	.21084	-.43517i	.13598	.39420i	.29074	-.02190i
1.8	.20310	-.37206i	.07454	.37900i	.28841	-.01409i
2.0	.18569	-.31087i	.02257	.34281i	.27643	.00248i
2.2	.16124	-.25212i	-.01429	.29125i	.25027	.02082i
2.4	.13173	-.19573i	-.03407	.23096i	.21010	.03377i
2.6	.09867	-.14133i	-.03808	.16756i	.15938	.03683i
2.8	.06326	-.08844i	-.02980	.10450i	.10248	.02929i
3.0	.02644	-.03650i	-.01367	.04295i	.04279	.01359i
3.1	.00778	-.01071i	-.00408	.01259i	.01259	.00411i
π	0	0	0	0	0	0

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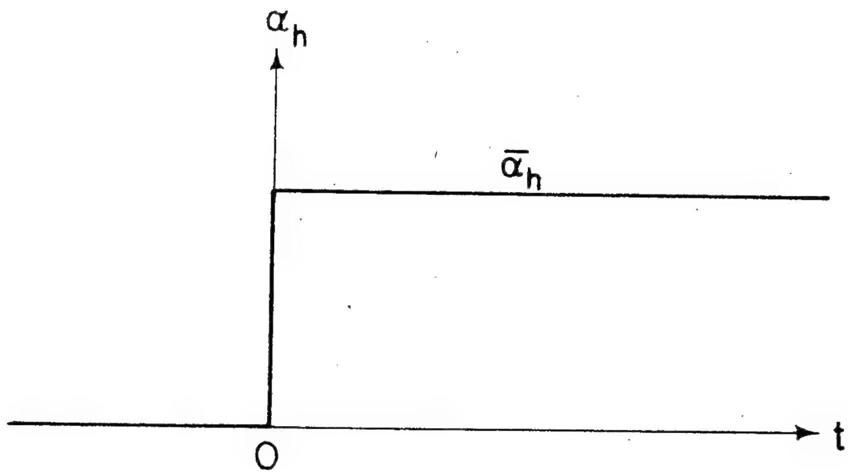
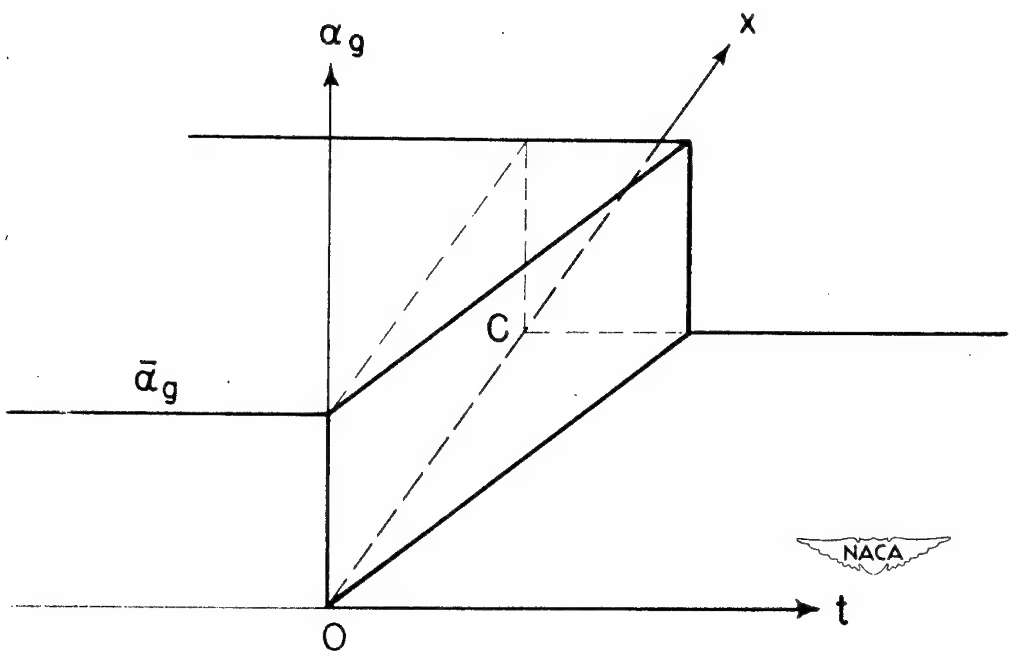


(a) $\frac{\dot{h}(t)}{U} = \alpha_h(t).$



(b) $\frac{\dot{g}(t)}{U} = \alpha_g(t).$

Figure 1.- Angle of attack as a function of time t for two cases: (a) Flapping wing and (b) wing meeting a vertical gust.

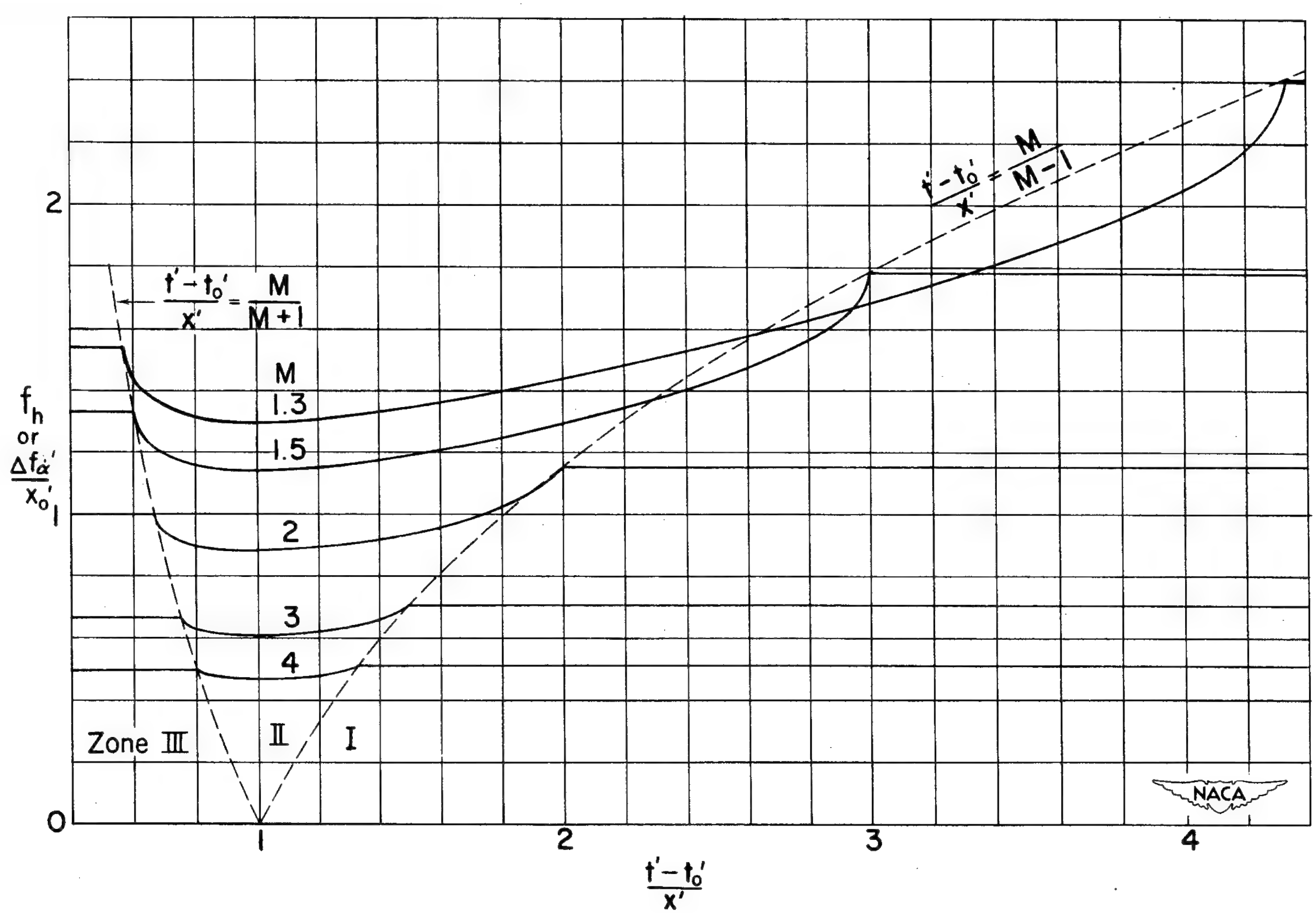


Figure 2.- Plot of f_h or $\frac{\Delta f_{\dot{\alpha}}'}{x_0'}$ as a function of $\frac{t' - t_0'}{x'}$ at various Mach numbers. $f_h = \frac{C_{ph}}{\bar{\alpha}_h}$; $\Delta f_{\dot{\alpha}}'$, contribution to $\frac{C_{p\dot{\alpha}}}{\dot{\alpha}_1 - \dot{\alpha}_0}$.

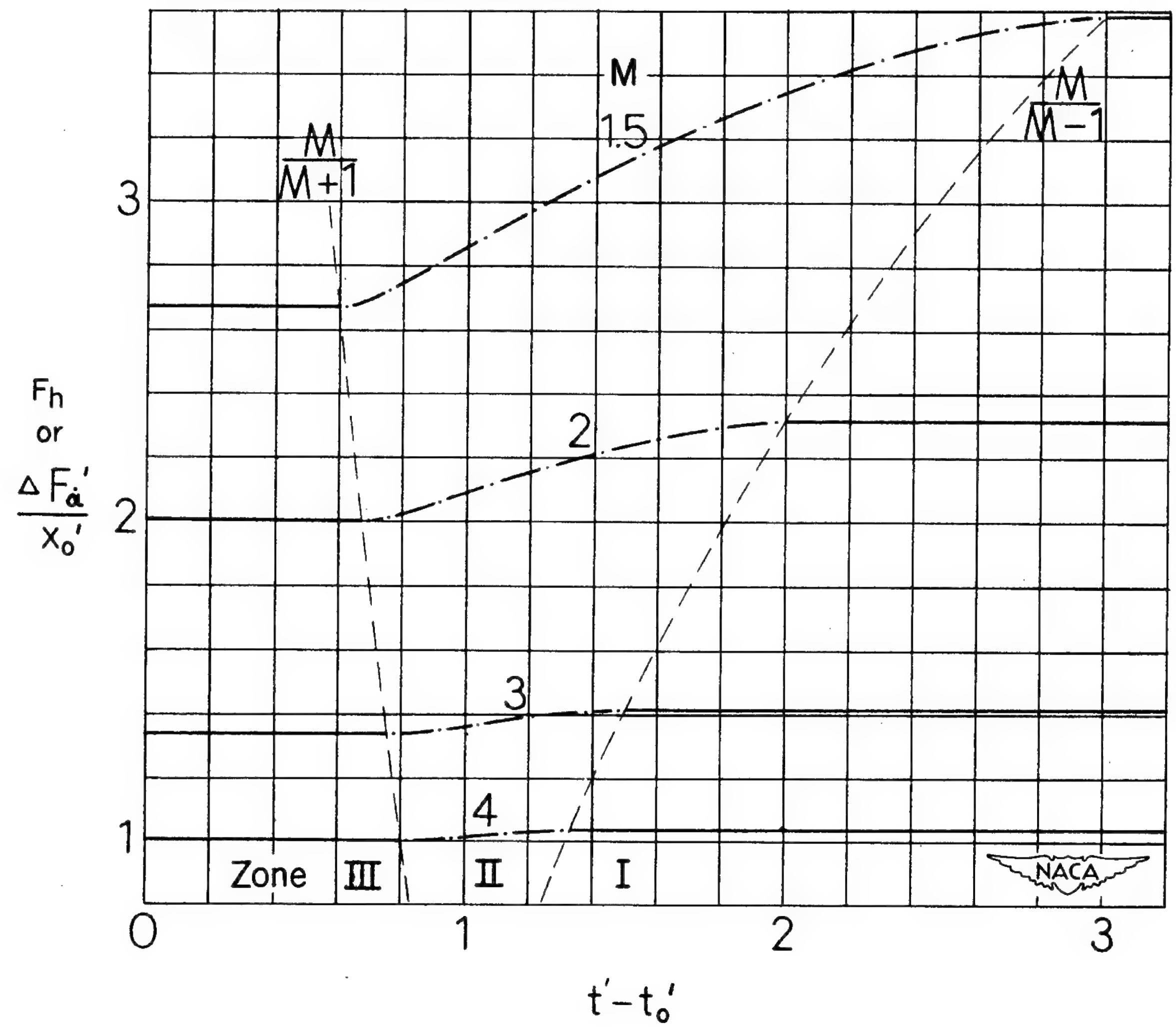


Figure 3.- Plot of F_h or $\frac{\Delta F_{\dot{\alpha}}'}{x_0'}$ as a function of $t' - t_0'$ at various Mach numbers. $F_h = \frac{C_{Lh}}{\bar{\alpha}_h}$; $\Delta F_{\dot{\alpha}}'$, contribution to $\frac{C_{L\dot{\alpha}}}{\dot{\alpha}_1 - \dot{\alpha}_0}$.

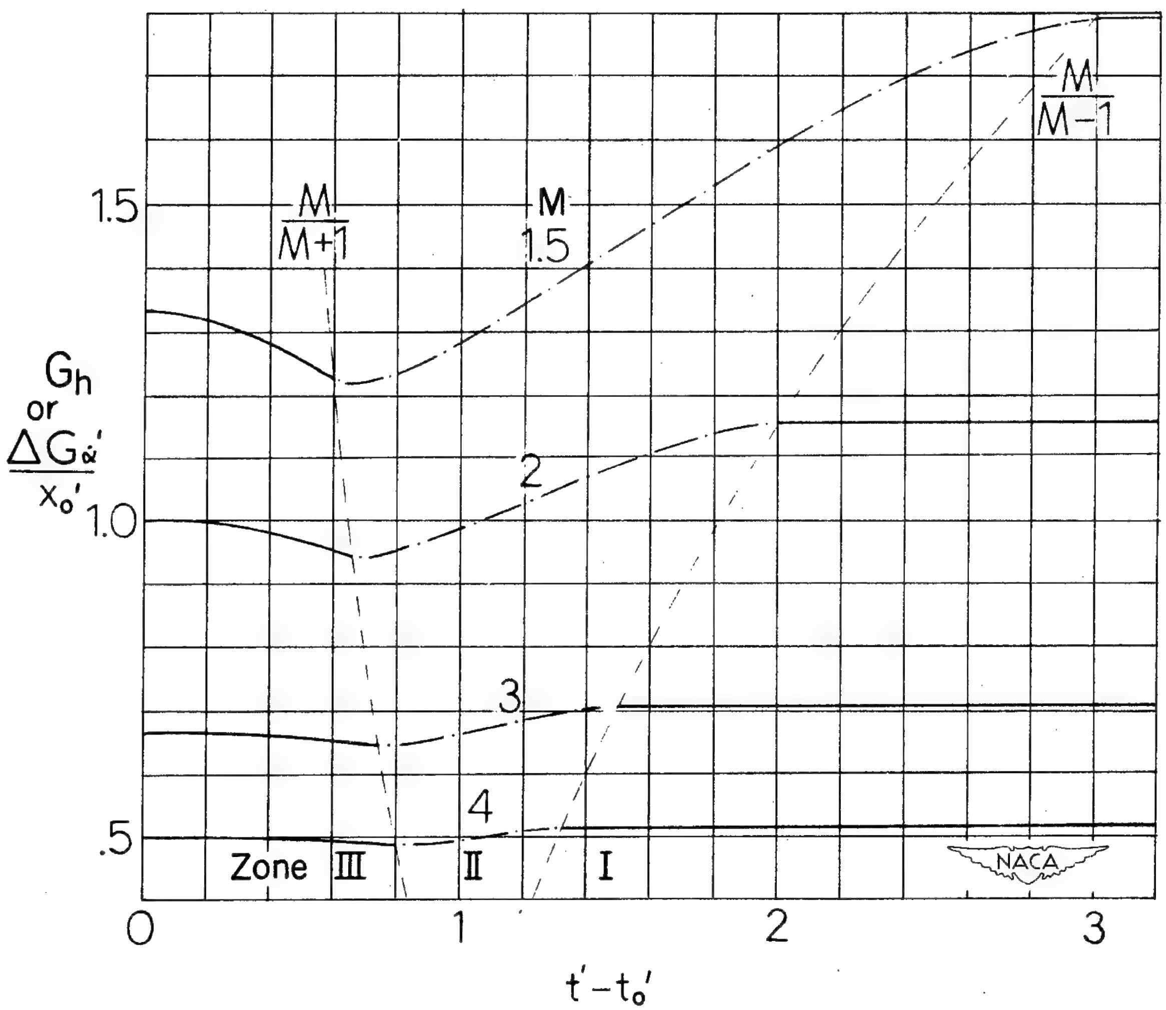


Figure 4.- Plot of G_h or $\frac{\Delta G_{\dot{\alpha}}'}{x_0'}$ as a function of $t' - t_0'$ at various Mach numbers. $G_h = \frac{C_{Mh}}{\bar{\alpha}_h}$; $\Delta G_{\dot{\alpha}}'$, contribution to $\frac{C_{M\dot{\alpha}}}{\dot{\alpha}_1 - \dot{\alpha}_0}$.

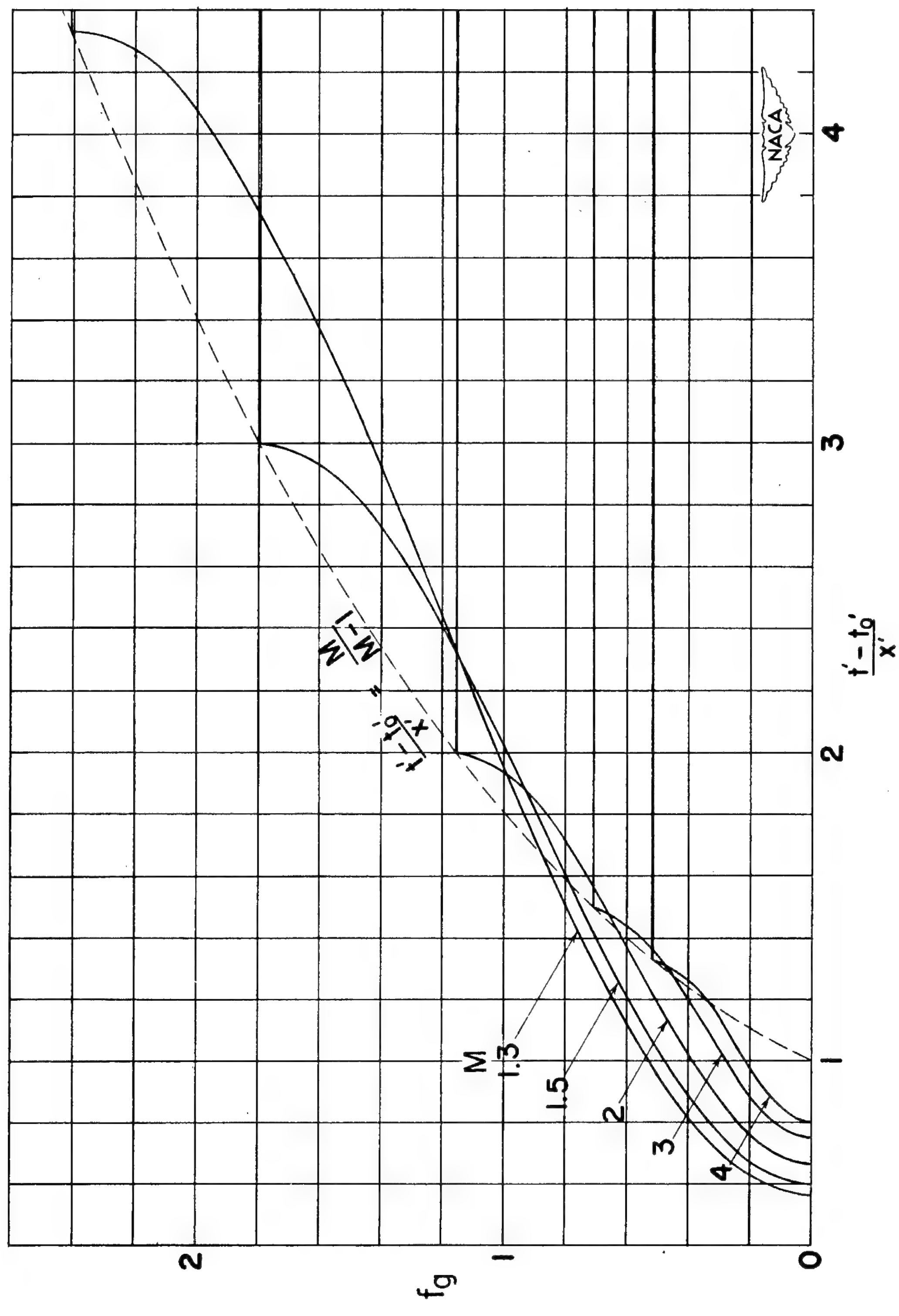


Figure 5.- Plot of f_g as a function of $\frac{t' - t_0'}{x'}$ at various Mach numbers. $f_g = \frac{C_{p_g}}{\bar{\alpha}_g}$.

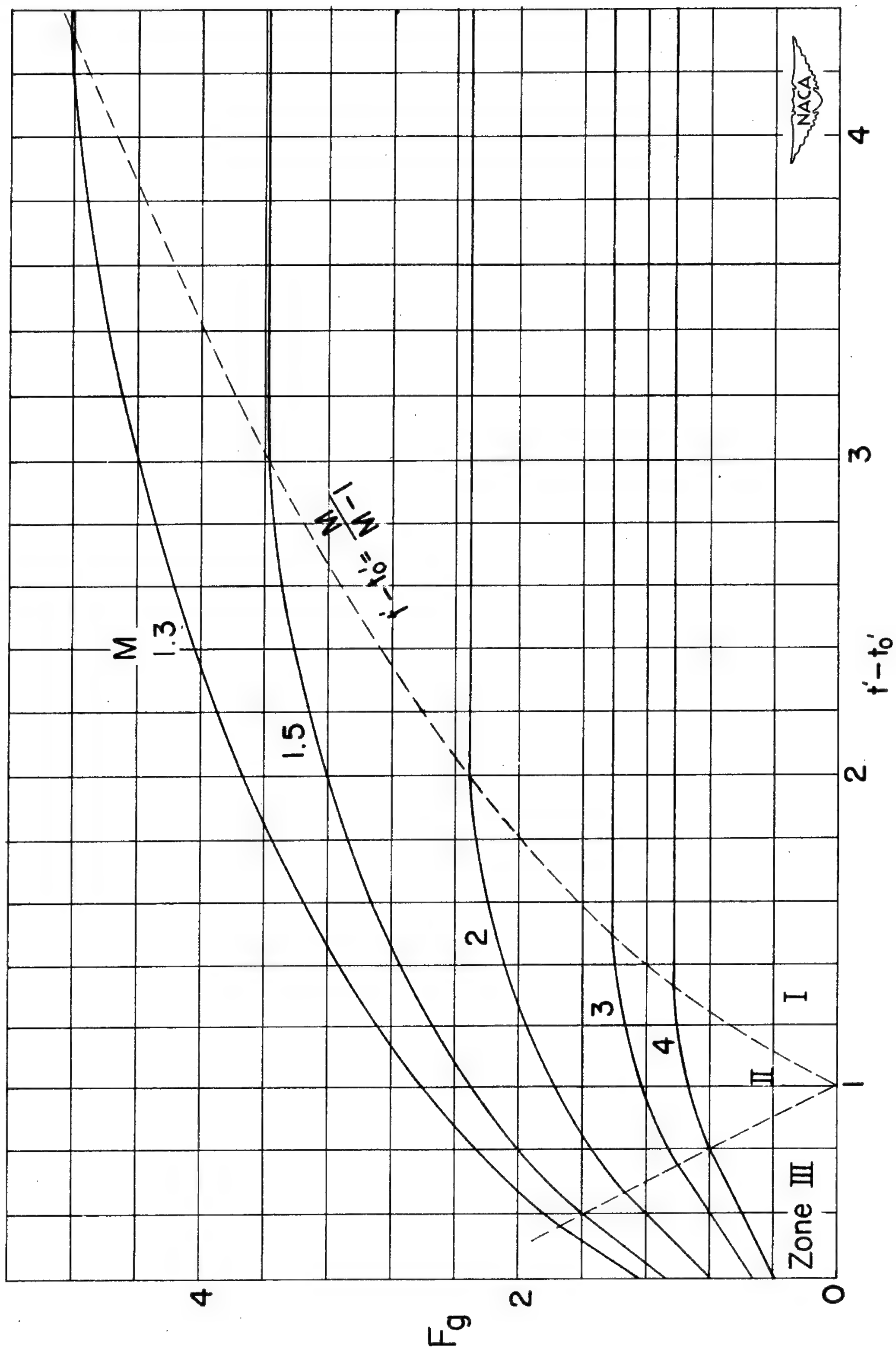


Figure 6.- Plot of F_g as a function of $t' - t_0'$ at various Mach numbers. $F_g = \frac{C_{L_g}}{\bar{\alpha}_g}$.

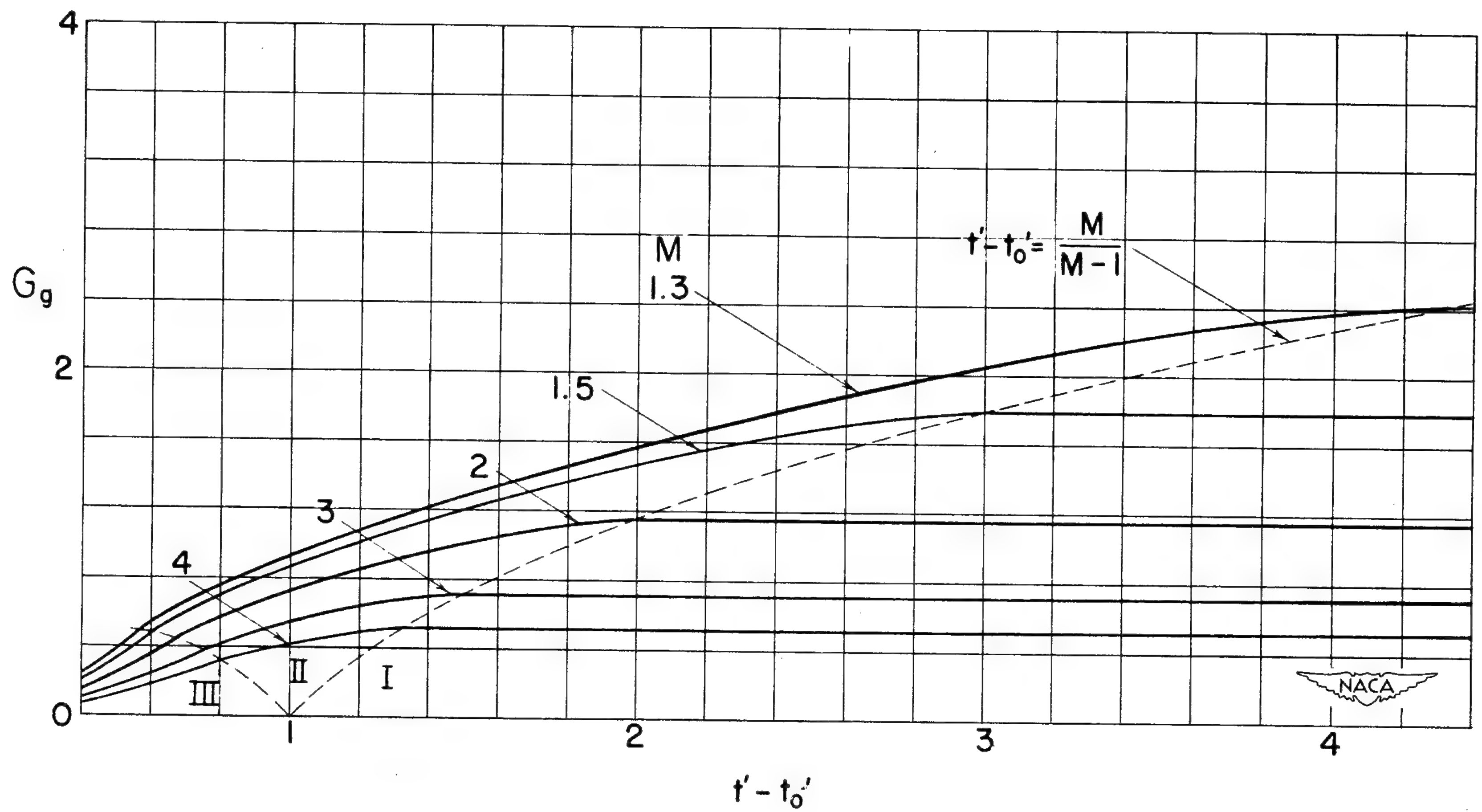


Figure 7.- Plot of G_g as a function of $t' - t_0'$ at various Mach numbers. $G_g = \frac{C_{M_g}}{\bar{\alpha}_g}$.

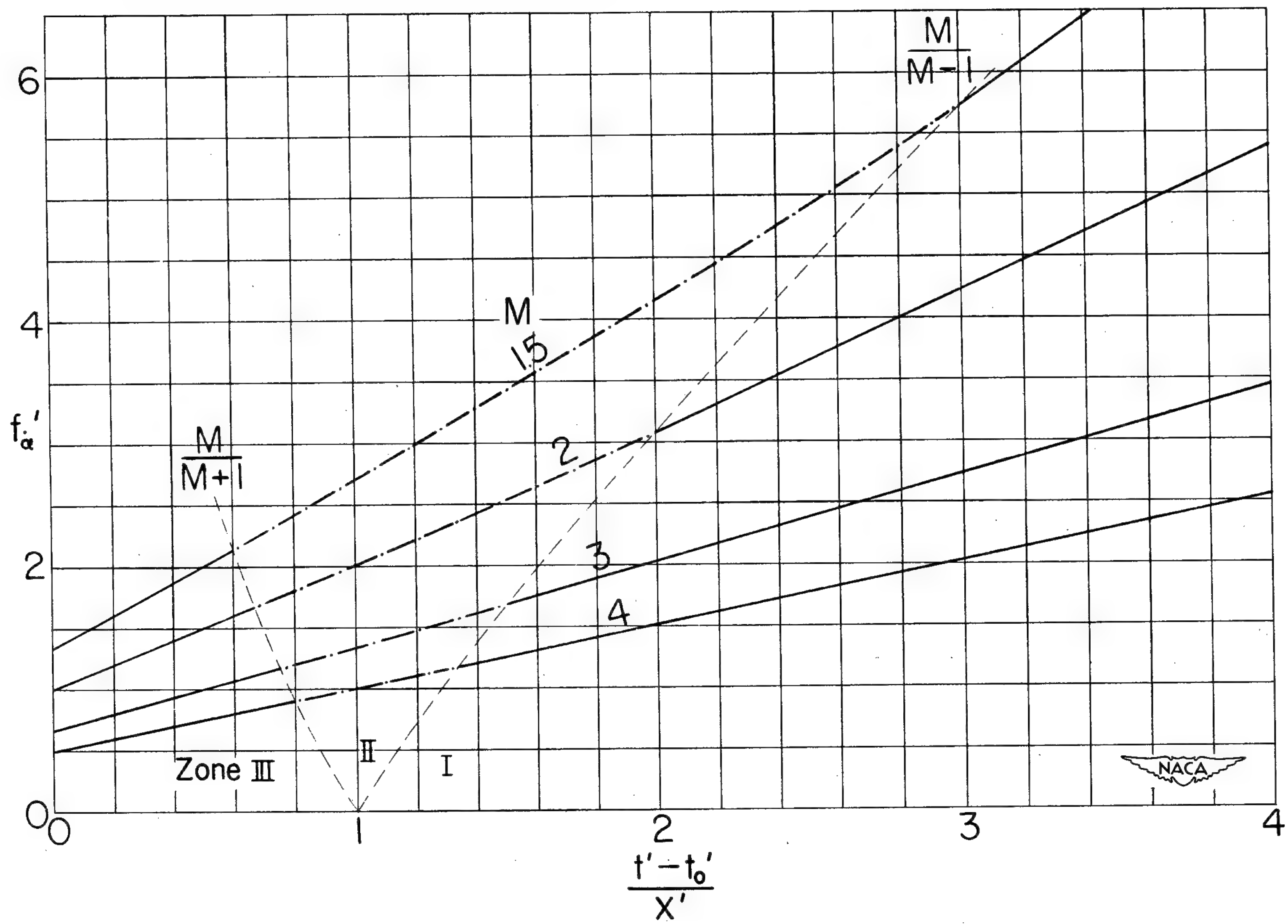


Figure 8.- Plot of $f_{\dot{\alpha}}'$ for $x_0' = 0$ as a function of $\frac{t' - t_0'}{x'}$ at various Mach numbers.

$$f_{\dot{\alpha}}' = \frac{C_{p\dot{\alpha}}}{x'\left(\dot{\alpha}_1 - \dot{\alpha}_0\right)}.$$

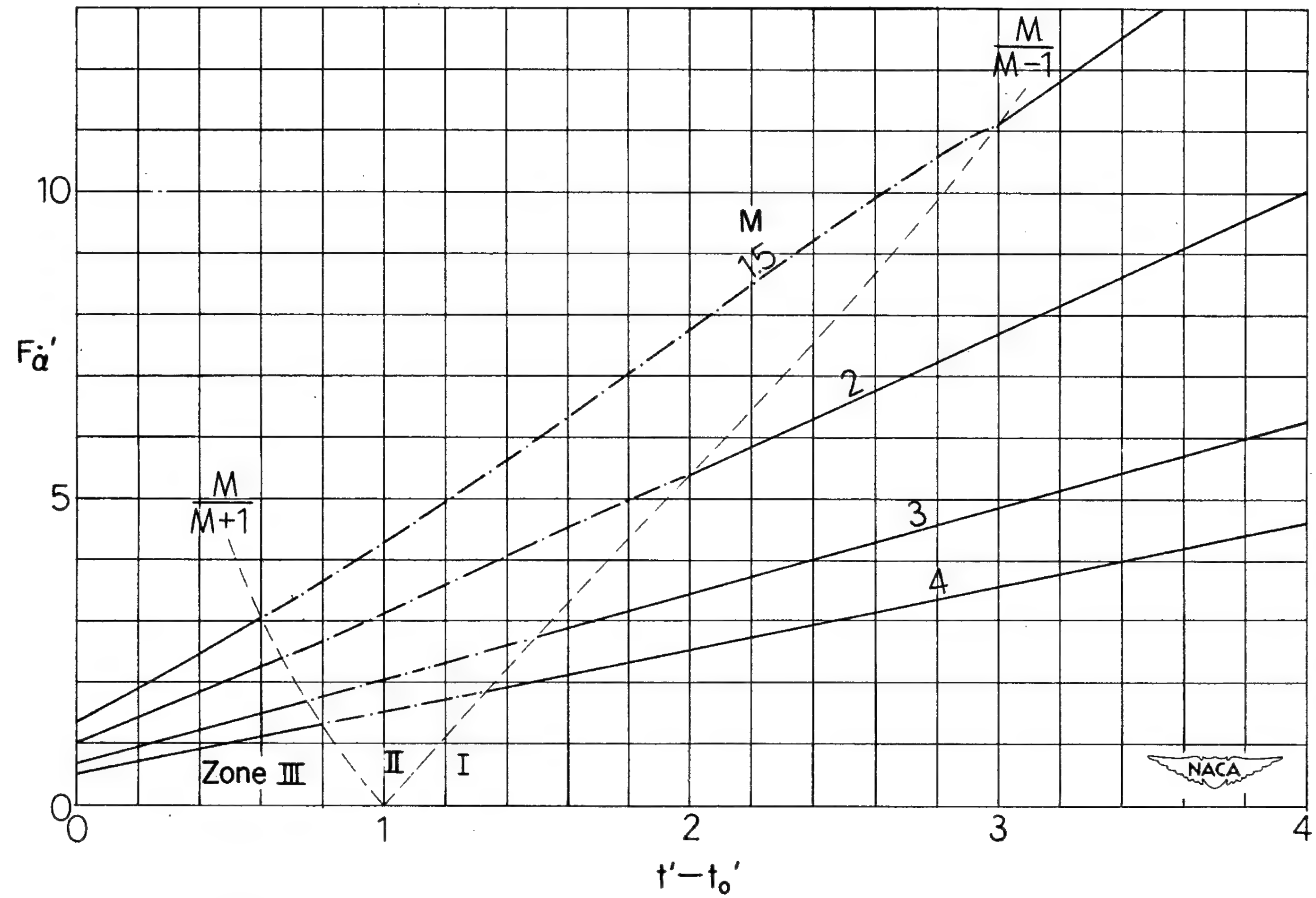


Figure 9.- Plot of $F_{\dot{\alpha}}'$ for $x_0' = 0$ as a function of $t' - t_0'$ at various Mach numbers.

$$F_{\dot{\alpha}}' = \frac{C_{L_{\dot{\alpha}}}(t' - t_0')}{\dot{\alpha}_1 - \dot{\alpha}_0}.$$

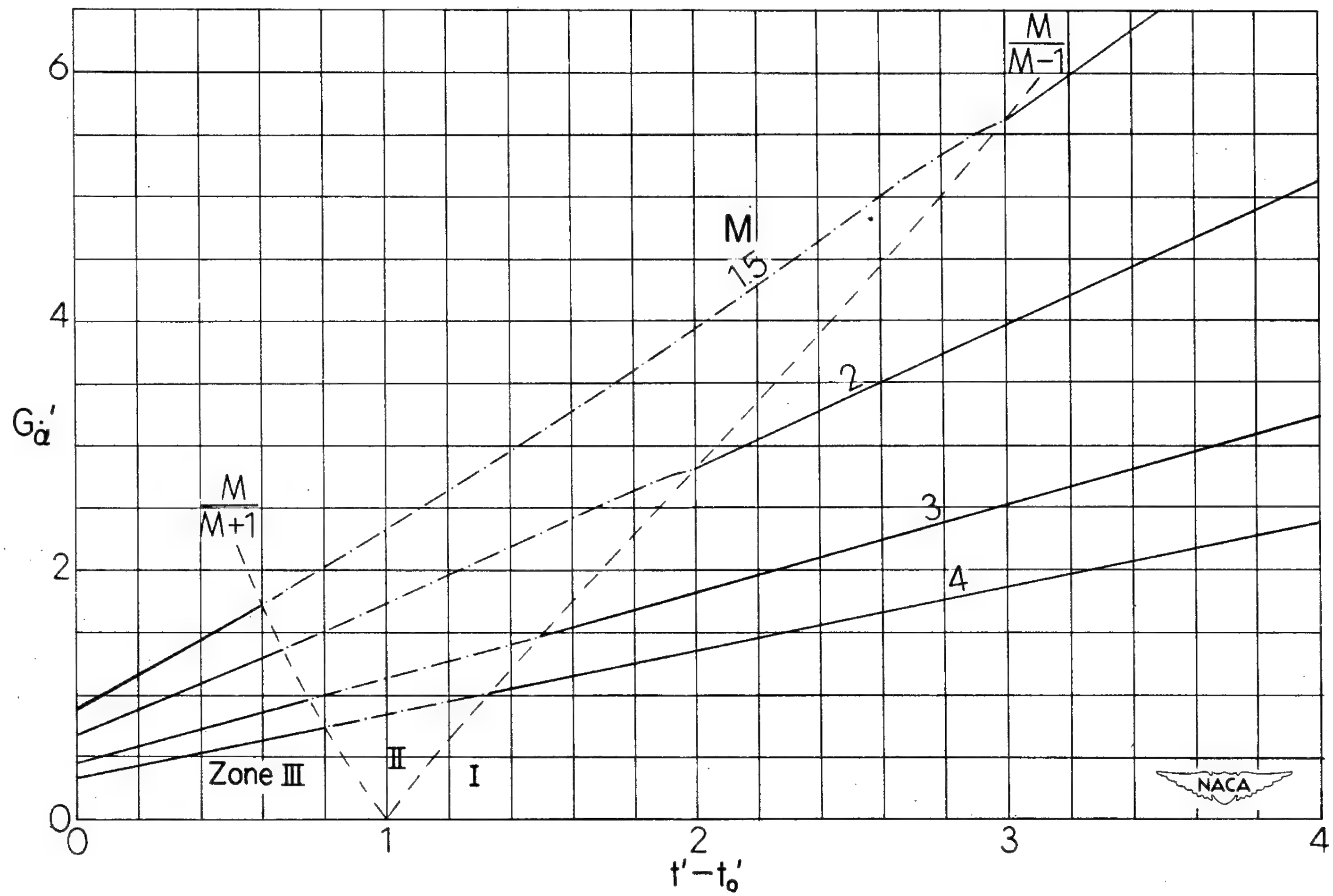


Figure 10.- Plot of $G_{\dot{\alpha}}'$ for $x_0' = 0$ as a function of $t' - t_0'$ at various Mach numbers.

$$G_{\dot{\alpha}}' = \frac{C_{M\dot{\alpha}}(t' - t_0')}{\dot{\alpha}_1 - \dot{\alpha}_0}.$$

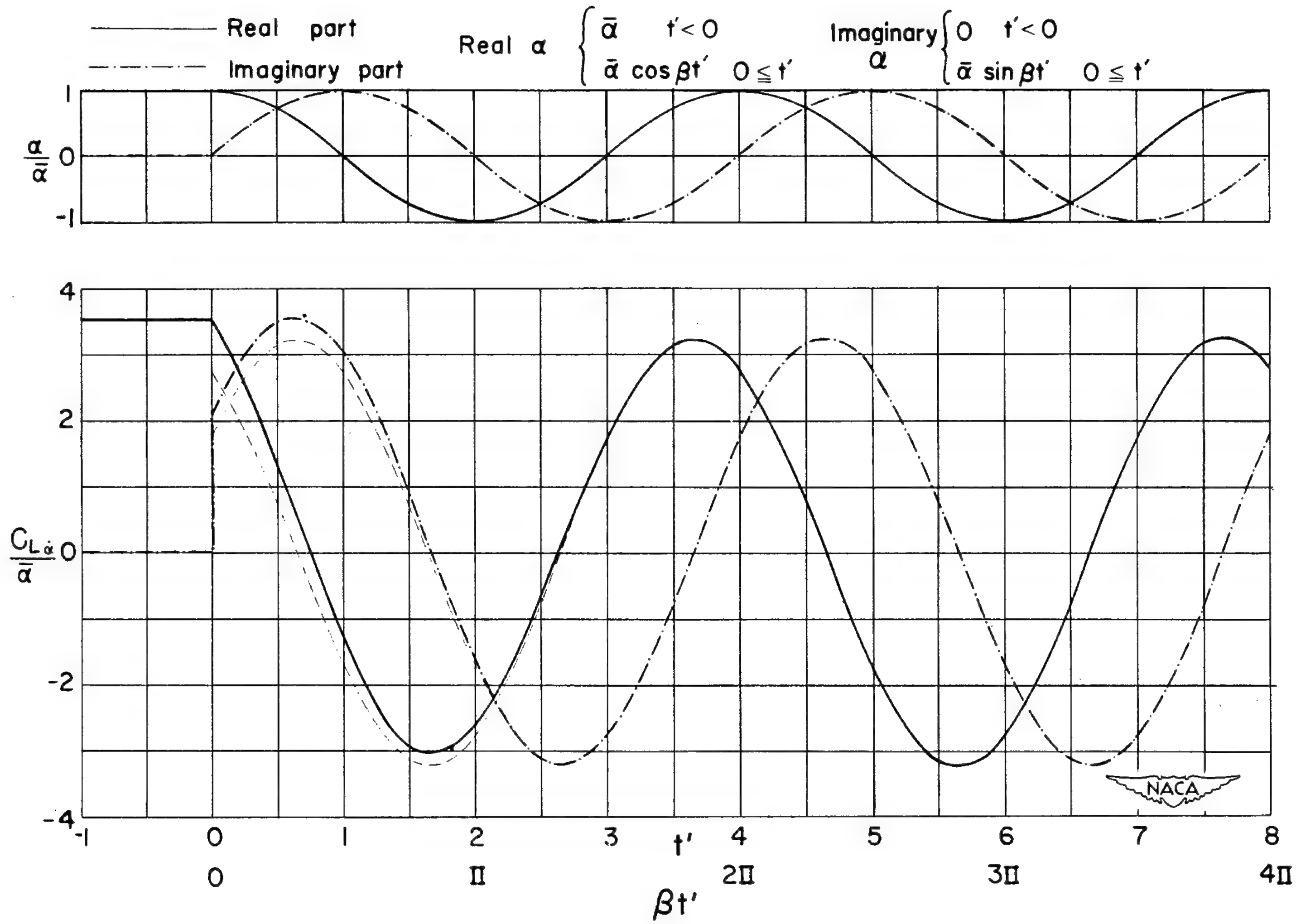


Figure 11.- Plots of $\alpha/\bar{\alpha}$ and $C_{L\dot{\alpha}}/\bar{\alpha}$ against t'(or $\beta t'$) for the harmonically oscillating angle of attack. $M = 1.5$; $\beta = \pi/2$; $t' = \frac{tU}{C}$.

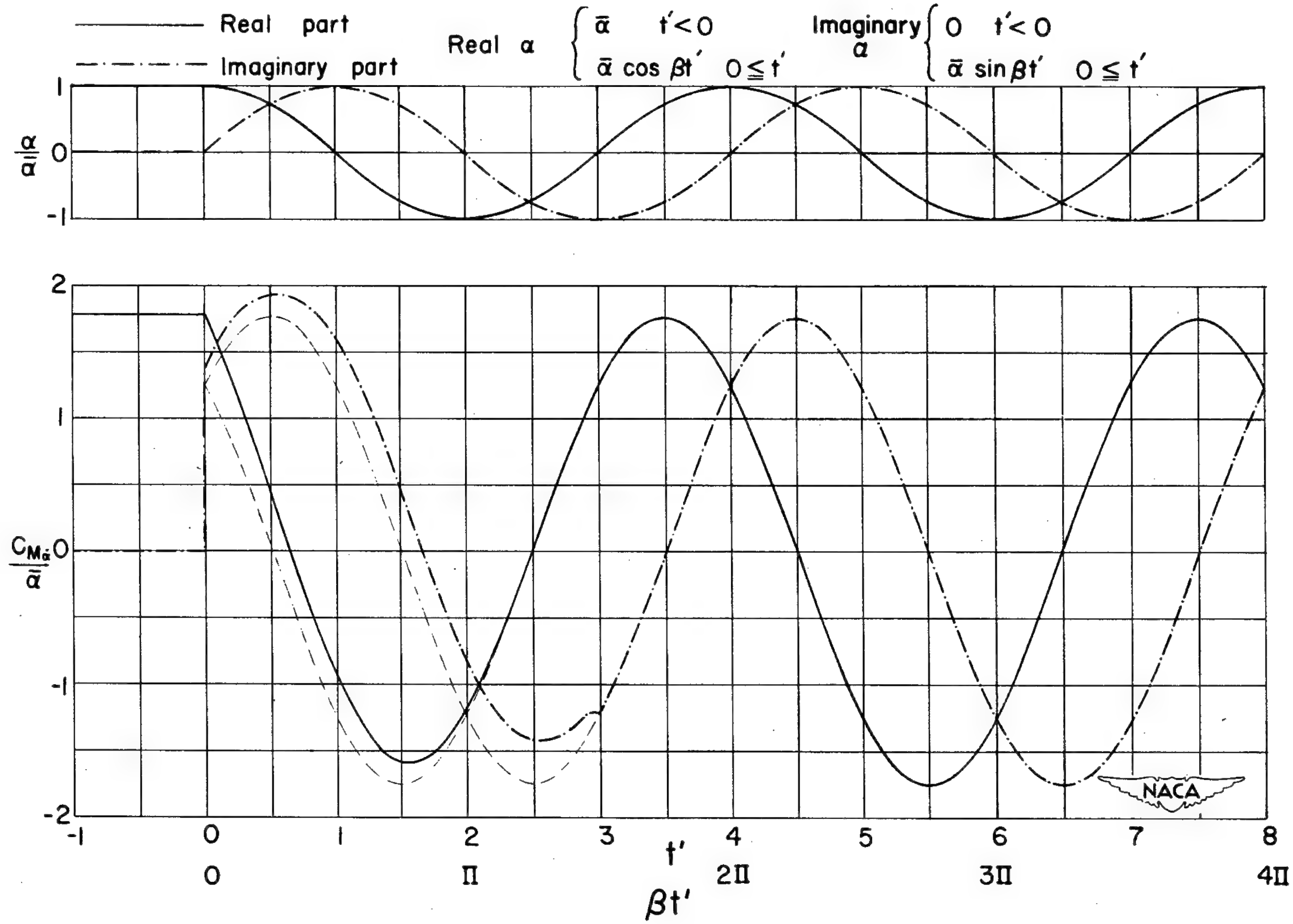


Figure 12.- Plots of $\alpha/\bar{\alpha}$ and $C_{M\dot{\alpha}}/\bar{\alpha}$ against t'(or $\beta t'$) for the harmonically oscillating angle of attack. $M = 1.5$; $\beta = \pi/2$; $t' = \frac{tU}{C}$.

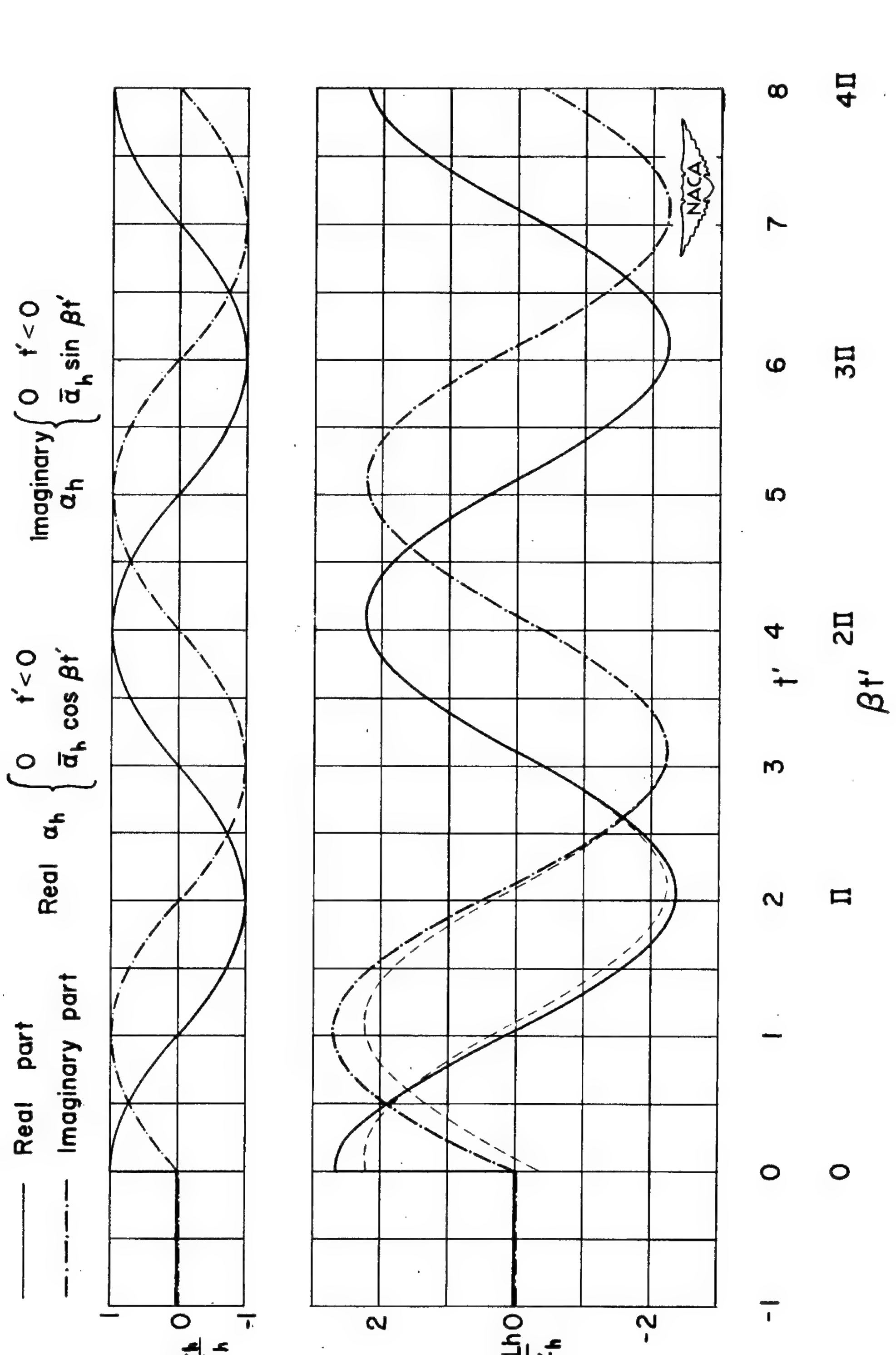


Figure 13.- Plots of $\alpha_h/\bar{\alpha}_h$ and $C_{Lh}/\bar{\alpha}_h$ against t'(or $\beta t'$) for the harmonically oscillating flapping airfoil. $M = 1.5$; $\beta = \pi/2$; $t' = \frac{tU}{C}$.

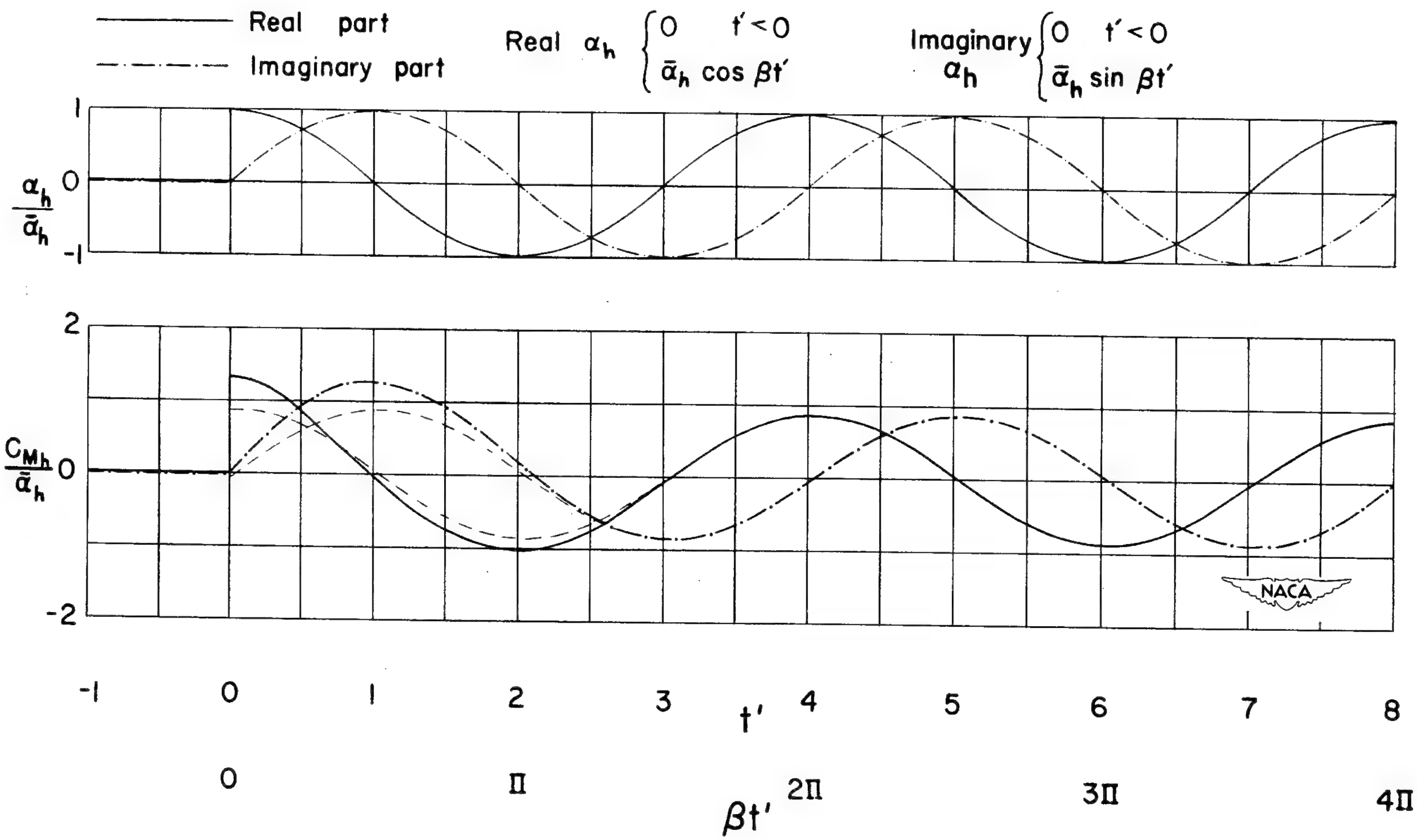


Figure 14.- Plots of $\alpha_h/\bar{\alpha}_h$ and $C_{Mh}/\bar{\alpha}_h$ against t'(or $\beta t'$) for the harmonically oscillating flapping airfoil. $M = 1.5$; $\beta = \pi/2$; $t' = \frac{tU}{C}$.

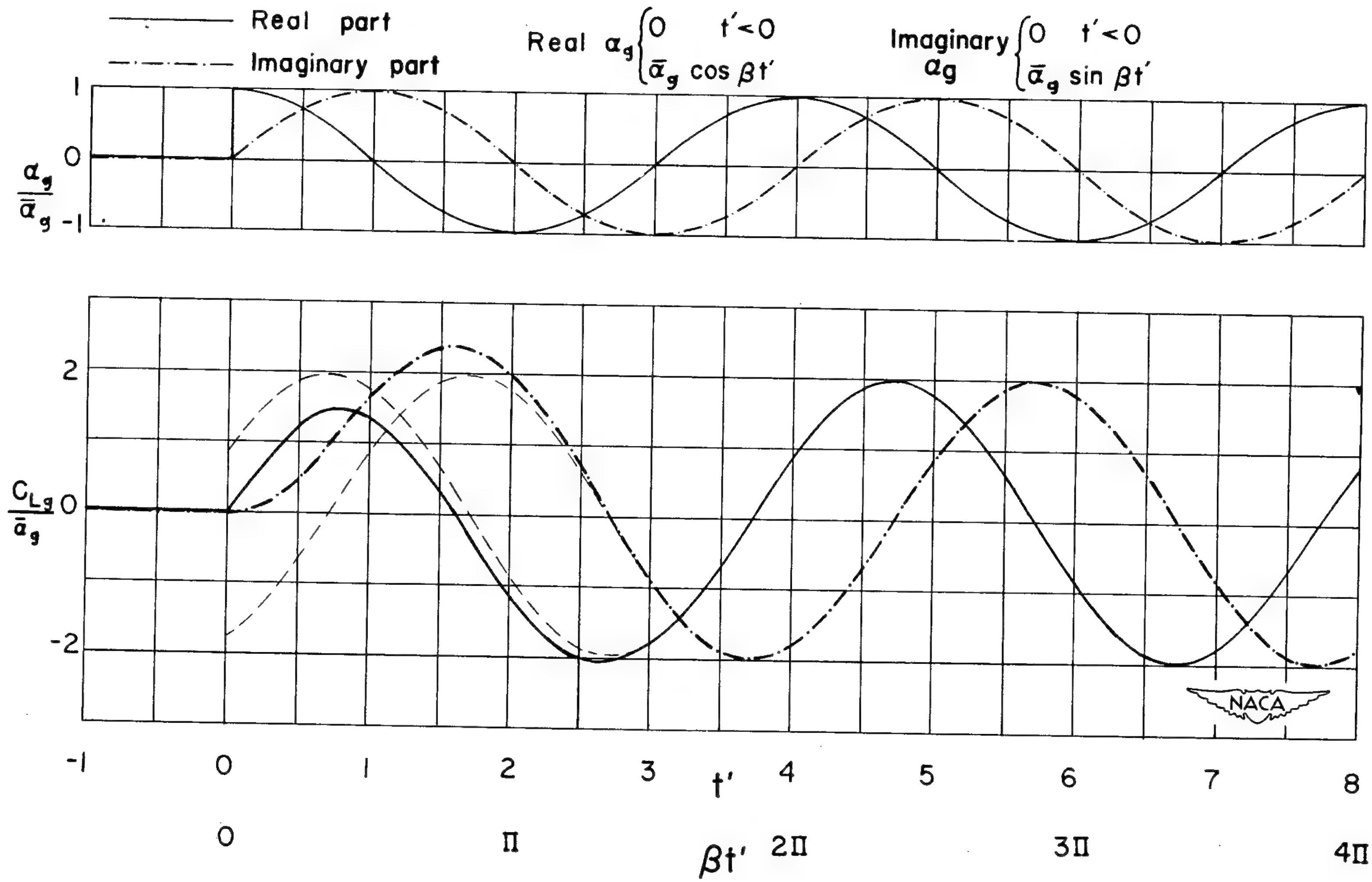


Figure 15.- Plots of $\alpha_g/\bar{\alpha}_g$ and $C_{Lg}/\bar{\alpha}_g$ against t'(or $\beta t'$) for the harmonically oscillating vertical gust. $M = 1.5$; $\beta = \pi/2$; $t' = \frac{tU}{C}$.

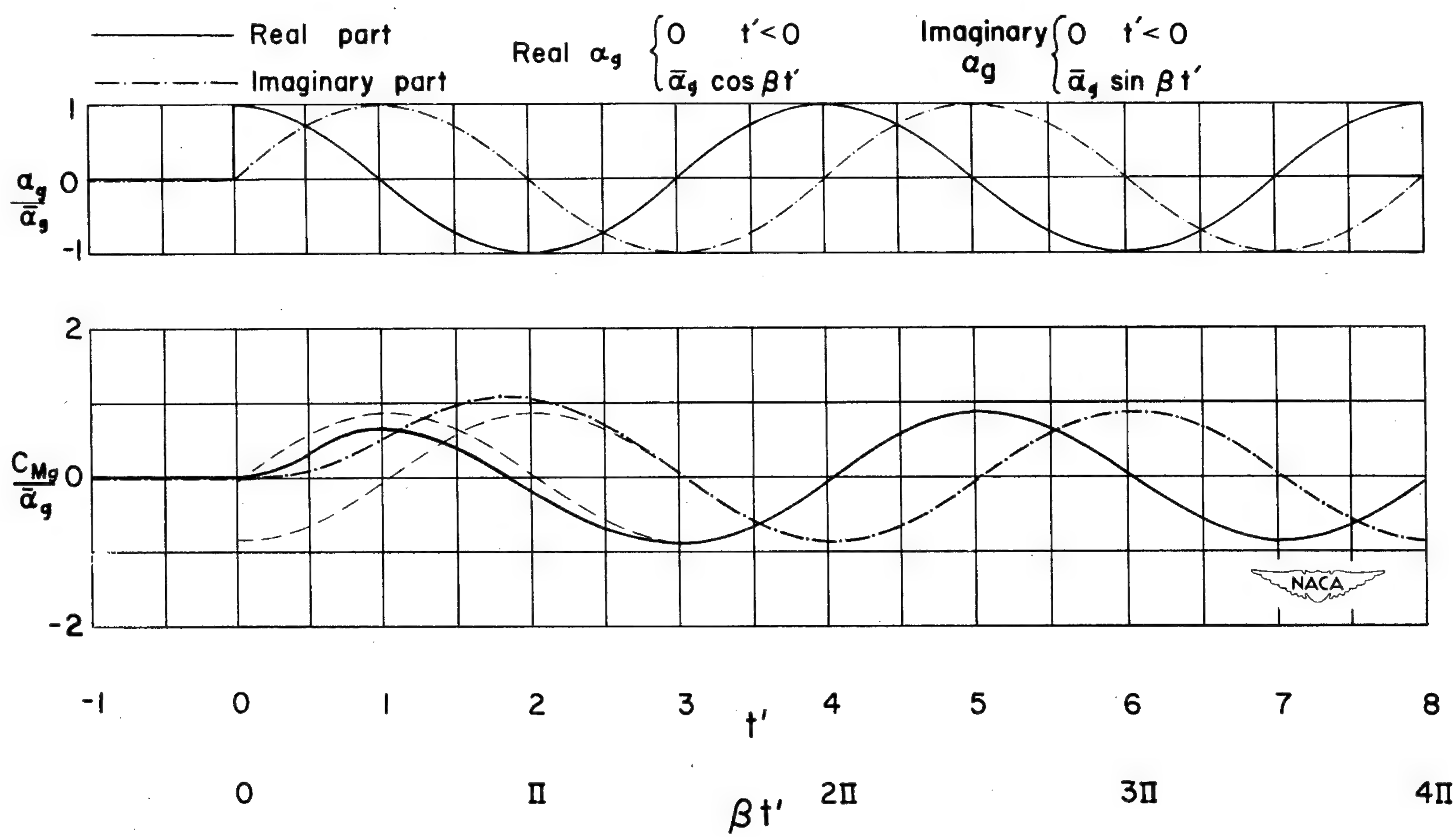


Figure 16.- Plots of $\alpha_g/\bar{\alpha}_g$ and $C_{Mg}/\bar{\alpha}_g$ against t'(or $\beta t'$) for the harmonically oscillating vertical gust. $M = 1.5$; $\beta = \pi/2$; $t' = \frac{tU}{C}$.

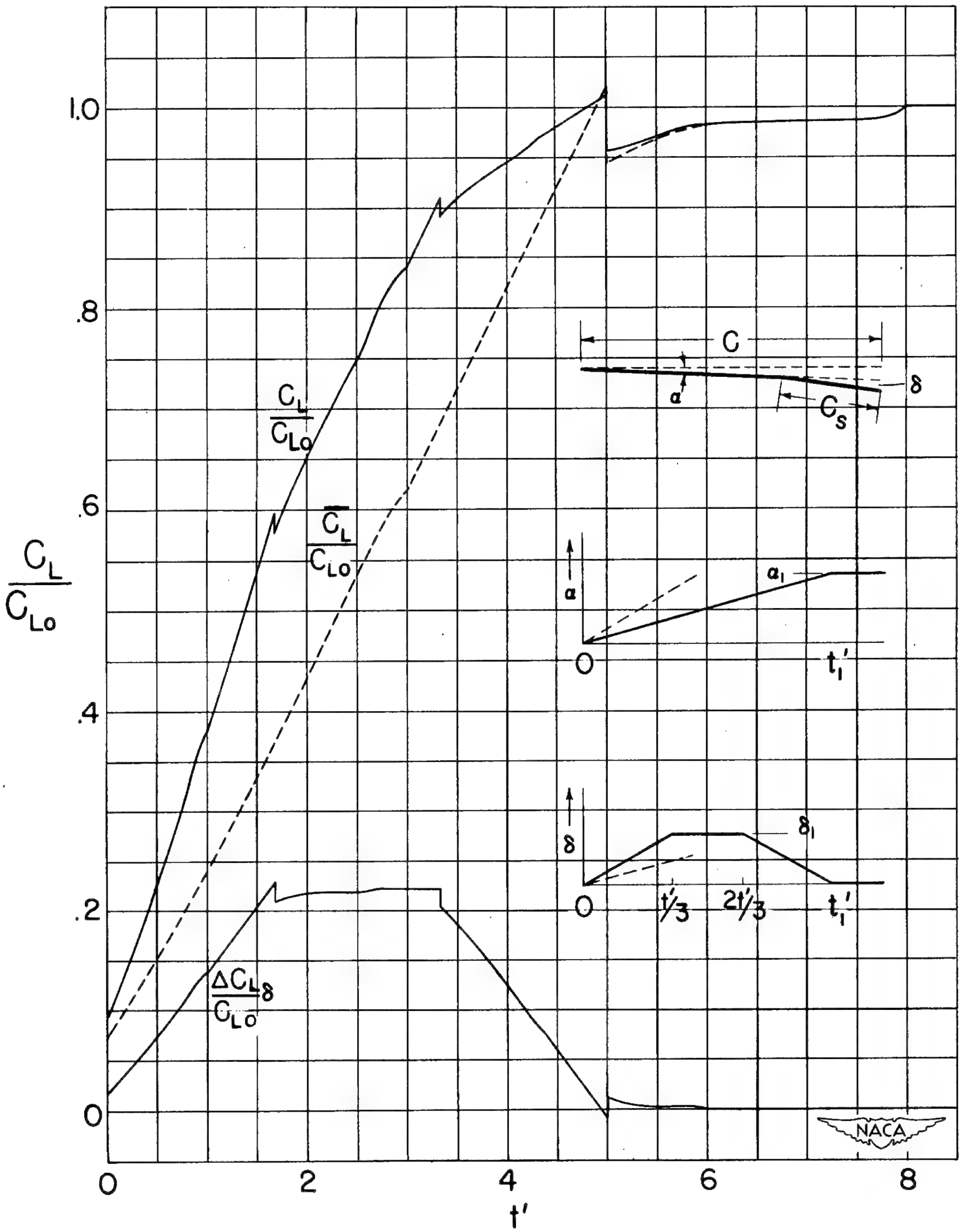


Figure 17.- Distribution of $\dot{\alpha}(t')$ and $\dot{\delta}(t')$ against t' and geometry of airfoil and control surface. Contribution of control surface to total lift shown by curve at bottom of figure. $\overline{C}_L(t')$ shown by dotted curve (from reference 1). $t_1' = 5$.

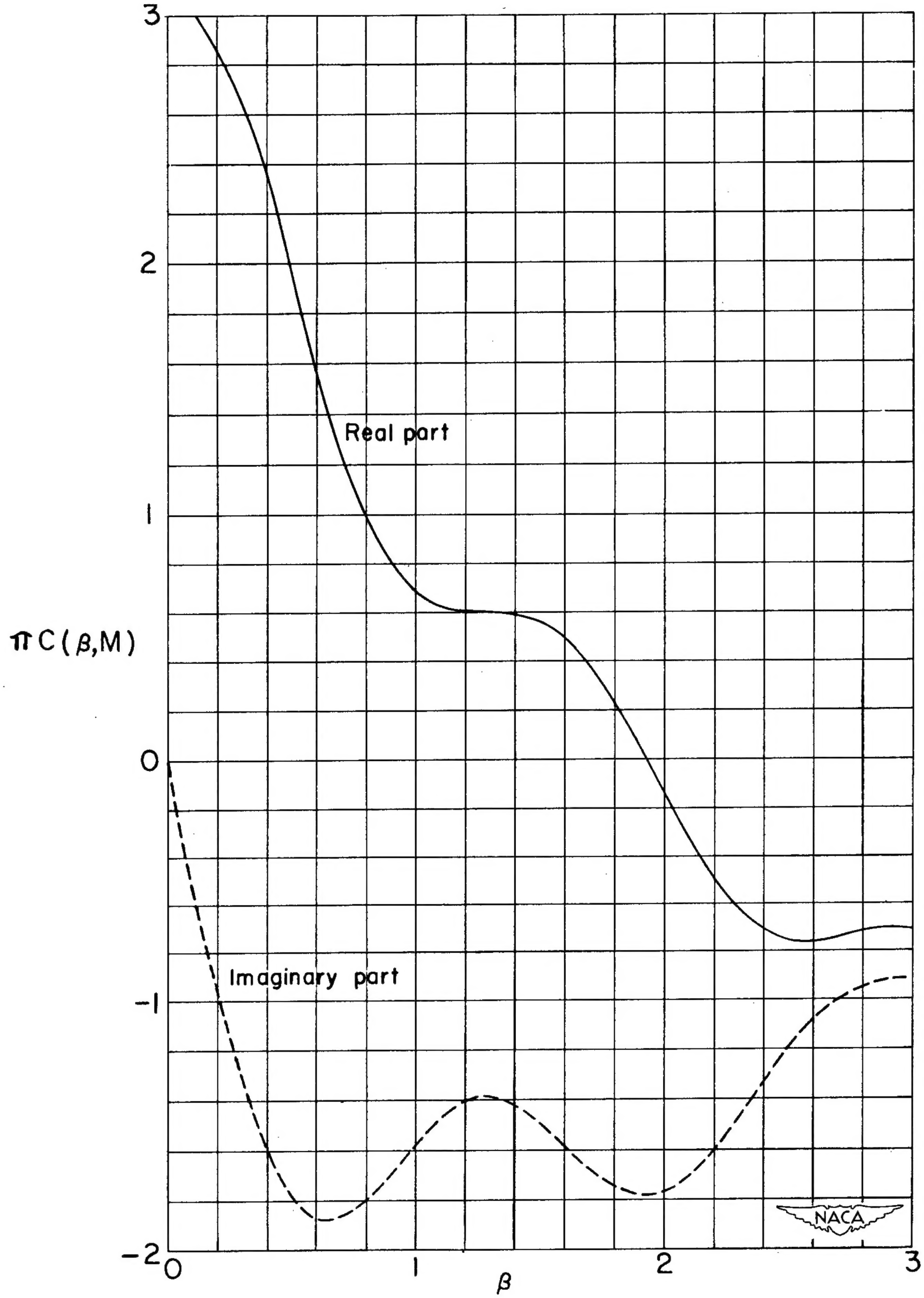


Figure 18.- Complete C-function $C(\beta,M)$ against β for $M = 1.3$.

$$C(\beta,M) = \frac{1}{\pi}\int_0^{\pi} e^{-i\beta\frac{M}{M - \cos\theta}}\,d\theta;\ \beta = \frac{\omega C}{U}.$$

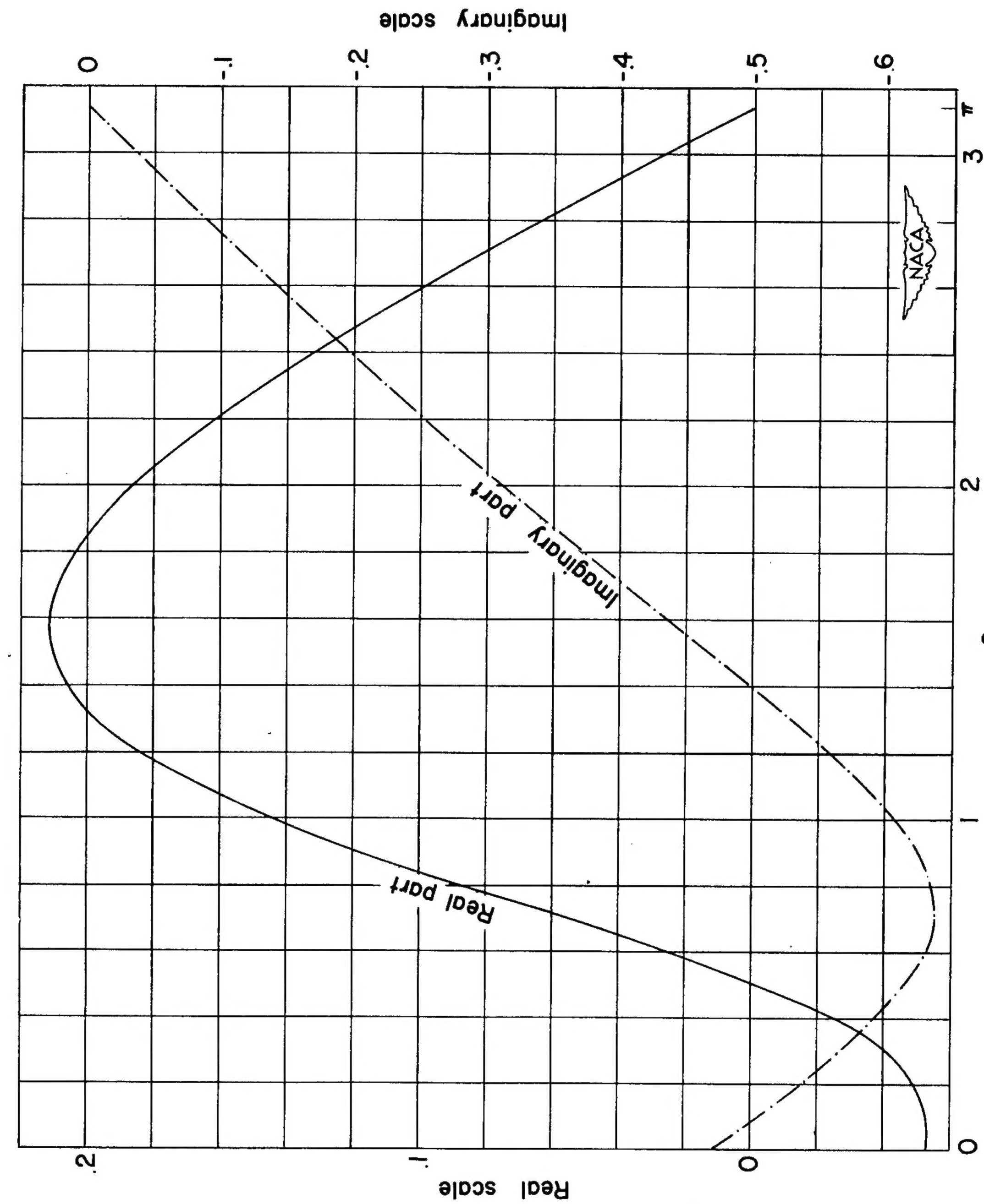


Figure 19.- Incomplete C-function $C(\beta,M;\ \theta_1)$ for $\beta = \pi/2$ and $M = 1.5$.

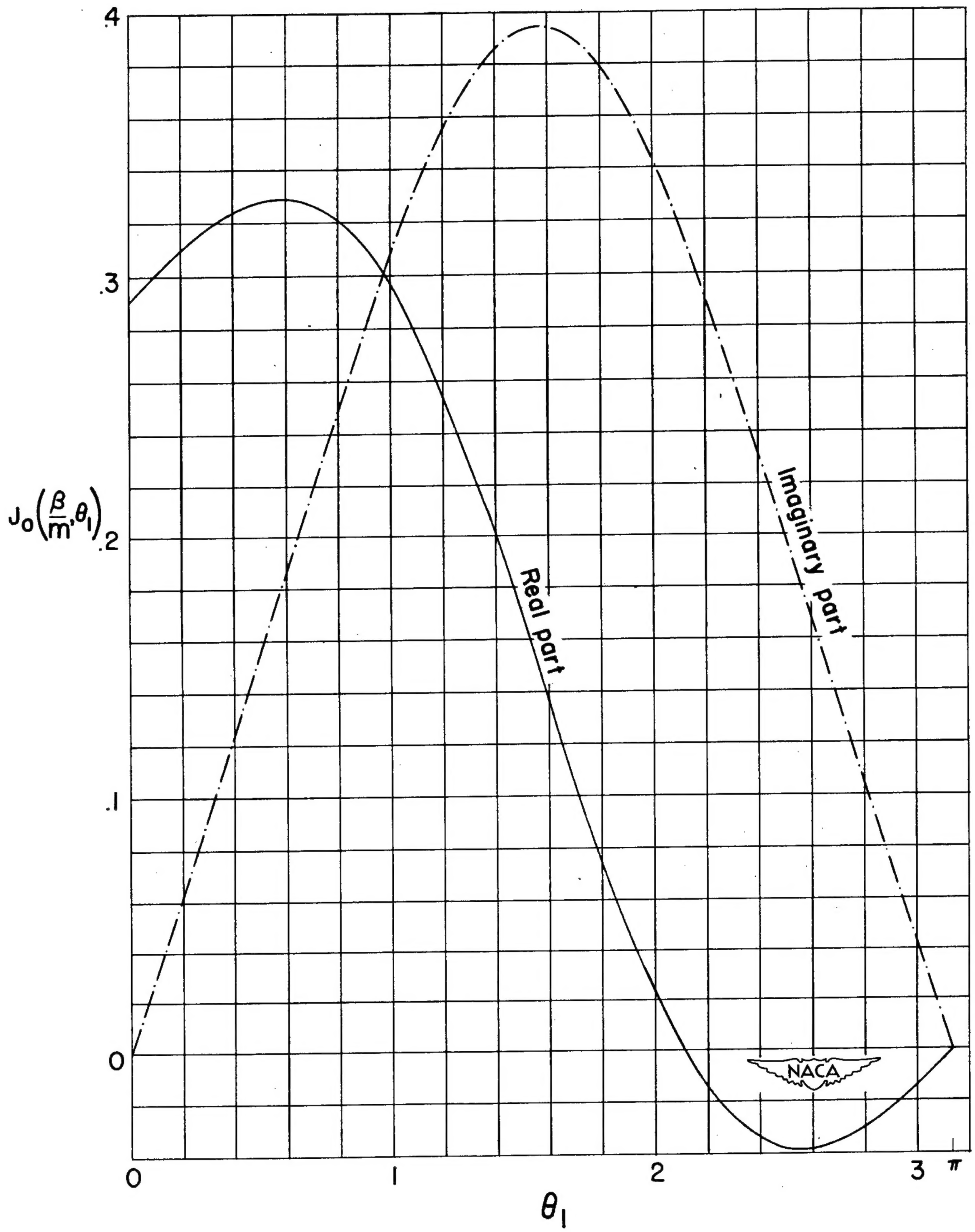


Figure 20.- Incomplete Bessel function of zero order $J_0\left(\frac{\beta}{m},\theta_1\right)$ for $M = 1.5$ and $\beta = \pi/2$.

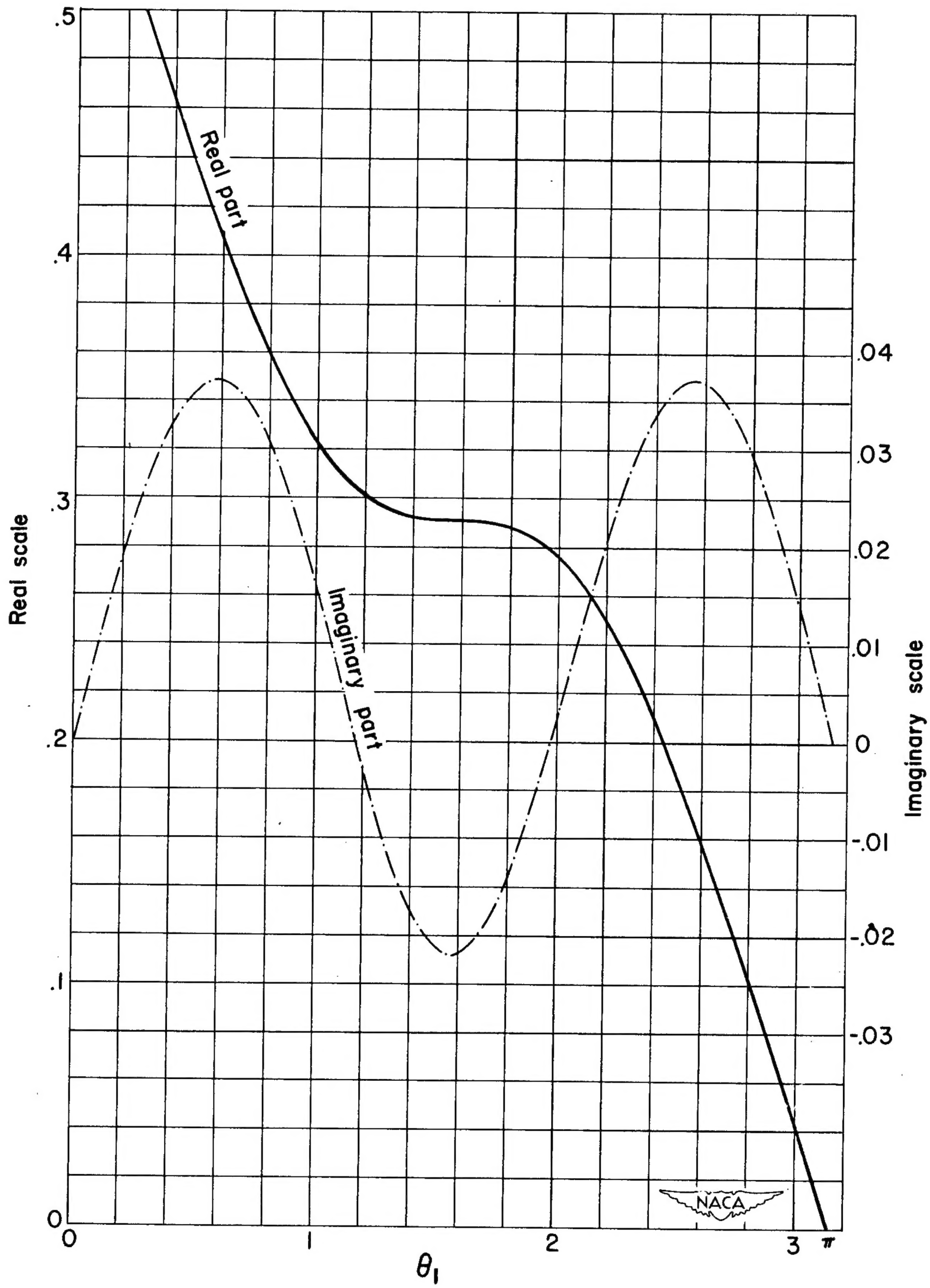


Figure 21.- Incomplete Bessel function of first order $J_1\left(\frac{\beta}{m},\theta_1\right)$ for $M = 1.5$ and $\beta = \pi/2$.